Fibred coarse embeddability of box spaces and proper isometric affine actions on L^p spaces

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Abstract

We show the necessary part of the following theorem : a finitely generated, residually finite group has property PL^p (i.e. it admits a proper isometric affine action on some L^p space) if, and only if, one (or equivalently, all) of its box spaces admits a fibred coarse embedding into some L^p space. We also prove that coarse embeddability of a box space of a group into a L^p space implies property PL^p for this group.

1 Introduction

The notion of fibred coarse embeddings into Hilbert space, which generalizes the notion of coarse embeddings, has been introduced by Chen, Wang and Yu in [CWY13] to provide a tool for the study of the maximal Baum-Connes conjecture. They proved in this paper that any metric space with bounded geometry admitting a fibred coarse embedding into a Hilbert space satisfies the maximal coarse Baum-Connes conjecture. In [CWW13], Chen, Wang and Wang characterized the Haagerup property in terms of fibred coarse embedding into Hilbert space : in fact, they showed that a finitely generated, residually finite group has the Haagerup property if, and only if, one of its box space admits a fibred coarse embedding into a Hilbert space. The goal of this note is to extend this result to the class of L^p spaces (for a fixed $p \geq 1$).

Theorem 1.1. *Let Γ be a finitely generated, residually finite group, $(\Gamma_i)_{i \in N^*}$ be a nested sequence of finite index normal subgroups of Γ with trivial intersection and $1 \leq p \leq \infty$. Then Γ has property PL^p if, and only if, the box space $\square_{\{\Gamma_i\}}\Gamma$ admits a fibred coarse embedding into a L^p space.*

We also prove the following proposition which extends to L^p spaces a result of Roe in the setting of Hilbert spaces (see [Roe03]).

Proposition 1.2. *Let Γ be a finitely generated, residually finite group, $(\Gamma_i)_{i \in N^*}$ be a nested sequence of finite index normal subgroups of Γ with trivial intersection and $1 \leq p \leq \infty$. If the box space $\square_{\{\Gamma_i\}}\Gamma$ admits a coarse embedding into a L^p space, then Γ has property PL^p.*

Theorem 1.1 and Proposition 1.2 can be stated for other classes of Banach spaces instead of L^p spaces. In fact, the proof of the necessary condition (see Proposition 3.4) and the proof of Proposition 1.2 only uses the fact that the class of L^p spaces (for a fixed $1 \leq p < \infty$) is a class $\mathcal{B}$ of Banach spaces satisfying the following properties :

1. $\mathcal{B}$ is closed under taking some particular normed finite powers i.e. :
   for every $n \in \mathbb{N}^*$ and every $B \in \mathcal{B}$, there exists a norm N on $\mathbb{R}^n$ such that :
   – there exists $c \geq 0$ such that, for all $K, K' \geq 0$ the n-cube $\{x \in \mathbb{R}^n \mid K \leq x_i \leq K'\}$ is contained in the annulus $\{x \in \mathbb{R}^n \mid cK \leq N(x) \leq cK'\}$ - or, in other words, for all $x \in \mathbb{R}^n$, if the components of x are controlled below by K and above by K' then so does $\frac{1}{c}N(x)$;

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– the Banach space B^n endowed with the norm $\|\cdot\| = N(\|\pi_1(\cdot)\|_B, ..., \|\pi_n(\cdot)\|_B)$ belongs to $\mathcal{B}$ (where π_i is the canonical projection of B^n on its i-th factor).

In the L^p case, for $n \in \mathbb{N}^*$, the norm of $\ell_p^n = \ell^p(\{1, ..., n\})$ fits, and $c = n^{\frac{1}{p}}$.

2. $\mathcal{B}$ is closed under ultraproducts (see Definition 3.2).
   In the L^p case, the stability by ultraproduct is a result due to Krivine (see [Kri67] Theorem 1 and its application p.17).

For a class of Banach spaces $\mathcal{B}$, property $P\mathcal{B}$ is an analog of the Haagerup property viewed with the Gromov's definition of a-T-menability (definition in terms of isometric affine actions, see [Gro93] or [CCJ+01]) where the class of Hilbert spaces is replaced by the class $\mathcal{B}$. One of the motivation in the study of this property is given by a result of Kasparov and Yu in [KY12] which asserts that groups admitting coarse embeddings into uniformly convex Banach spaces satisfy the Novikov conjecture (in particular, groups having property $P\mathcal{B}$ where $\mathcal{B}$ is a subclass of uniformly convex Banach spaces admit such embeddings).

An *isometric affine action* of a group Γ on a Banach space B is a morphism α of Γ into the group $\mathrm{Aff}(B) \cap \mathrm{Isom}(B)$ of affine isometric transformations of B; such an action can be characterized by the following decomposition :

$$\alpha(g)v = \pi(g)v + b(g), \text{ for all } g \in \Gamma, \ v \in B,$$

where π is an isometric representation of Γ on B and b is a 1-cocycle with respect to π i.e., for all $g, h \in \Gamma$, $b(gh) = \pi(g)b(h) + b(g)$.
The action α is said to be *proper* if $\|b(g)\|_B \underset{g\to\infty}{\longrightarrow} +\infty$.

Definition 1.3. *Let $\mathcal{B}$ be a class of Banach spaces. A (discrete) group Γ is said to have* property $P\mathcal{B}$ *if there exists a proper isometric affine action of Γ on some Banach space $B \in \mathcal{B}$.*

Many recent progress has been made in the study of isometric affine actions on Banach spaces, and more particularly in the case of L^p spaces for a fixed $1 \leq p \leq \infty$. Bader, Furman, Gelander, Monod studied the relationships between two different generalizations of Kazhdan's property (T), namely property FL^p and property (T_{L^p}) in [BFGM07]. On the other hand, property PL^p, also referred as a-FL^p-menability by some authors, is a strong negation of property FL^p. Examples of PL^p groups are given by [Yu05], where Yu proved that, for a discrete hyperbolic group Γ, there exists $2 \leq p_0 < \infty$ such that Γ has property PL^p for all $p \geq p_0$; or by [CTV08], where Cornulier, Tessera, Valette showed that the hyperbolic simple Lie group $\mathrm{Sp}(n, 1)$ has property PL^p for all $p > 4n + 2$. We give here an overview of what is known about the links between property PL^p and PL^q for various values of p and q :

(1)	Haagerup ($=PL^2$)	$\Rightarrow$	PL^p for all $0 < p \leq \infty$
(2)	PL^p for some $0 < p \leq 2$	$\Leftrightarrow$	Haagerup
(3)	PL^p for some $p > 2$	$\not\Rightarrow$	Haagerup
(4)	PL^p for some $p > 2$	$\underset{??}{\Rightarrow}$	PL^q for all $q > p$

Implication (1) was proved in [CMV04] by Cherix, Martin and Valette for countable discrete groups, using the notion of spaces with measured walls. Equivalence (2) follows from results of Delorme-Guichardet ([Gui72], [Del77]) and Akemann-Walter ([AW81]). See [CDH10] Corollary 1.5 and Remark 1.6 for proofs and discussions about (1) and (2) in the setting of second countable, locally compact groups.
Assertion (3) follows from the fact that a discrete hyperbolic group with property (T) fails the Haagerup property but has PL^p for some $p > 2$ by the result of Yu quoted before. We mention that assertion (4) is still an open question which appears in [CDH10], Question 1.8.

Concerning stability, property PL^p (for a fixed $p > 2$) is closed under taking closed subgroups, direct sums, amalgamated free products over finite subgroups (see [Pil15] and [Arn13] for proofs of this result with different approachs) but it is not stable by extension in general. However, using a construction of Cornulier, Stalder and Valette in [CSV12], the author showed in [Arn14] that property PL^p is closed under wreath product by Haagerup groups. We would like to mention that Haagerup property is stable by amenable extensions, but for property PL^p with $p > 2$, it remains an open problem.

Remark 1.4. *Notice that unlike in the Hilbert spaces case, property PL^p is no longer equivalent to property HL^p i.e. the existence of a C_0 representation on some L^p space which almost has invariant vectors. For instance, a discrete hyperbolic group with property (T) has property PL^p for some $p > 2$, but it also has property (T_{L^p}) for all $p \geq 1$ (see [BFGM07]) which is a strong negation of property HL^p.*

Definition 1.5. *Let Γ be a finitely generated, residually finite group and let $\Gamma_1 \trianglerighteq ... \trianglerighteq \Gamma_i \trianglerighteq ...$ be a nested sequence of finite index normal subgroups of Γ such that $\bigcap_{i=1}^{\infty} \Gamma_i = \{e\}$. The box space associated with the sequence $\{\Gamma_i\}_{i \in \mathbb{N}^*}$, denoted by $\square_{\{\Gamma_i\}}\Gamma$ or simply $\square\Gamma$, is the coarse disjoint union $\bigsqcup_{i=1}^{\infty} \Gamma/\Gamma_i$ of the finite quotient groups, i.e., the disjoint union where each quotient is endowed with the metric induced by the image of the generating set of Γ, and the distances between the identity elements of two successive quotients are chosen to be greater than the maximum of their diameters.*

There is a large spectrum of analytic properties of a group Γ which link to geometric properties of its box space $\square\Gamma$. As in [CWW13], we summarize here different correspondances :

Γ amenable	$\Leftrightarrow$	$\square\Gamma$ Property A
Γ Property (T)	$\Leftrightarrow$	$\square\Gamma$ geometric Property (T)
Γ Haagerup	$\Leftrightarrow$	$\square\Gamma$ fibred coarsely embeddable into Hilbert space
Γ Property PL^p	$\Leftrightarrow$	$\square\Gamma$ fibred coarsely embeddable into some L^p
Γ Property PL^p	$\Leftarrow$	$\square\Gamma$ coarsely embeddable into some L^p

The first equivalence was established by Roe in [Roe03] where Property A is a non-equivariant version of amenability defined by Yu ([Yu00]) which guarantees coarse embeddability into Hilbert spaces. The second one is due to Willett and Yu in [WY12] where they introduced the notion of geometric property (T). For a coarse disjoint union of finite graphs, geometric property (T) implies the property of being an expander. The third equivalence is the result of Chen, Wang and Wang ([CWW13]) mentioned in the introduction.
The last two assertions are proved in the present note. In [Roe03], Roe established the last implication in the Hilbert case ($p = 2$ case); and notice that the converse implication fails. In fact, on one hand, the free group on two generators has the Haagerup property, and on the other hand, it has property (τ) with respect to some sequences of finite index normal subgroups (see [Lub10]) : hence, the associated box spaces are expanders, which implies that they are not coarsely embaddable into Hilbert space.

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2 Fibred Coarse embeddings into Banach spaces

We recall here the notion of coarse embedding and the notion of fibred coarse embedding introduced in [CWY13] where the notion of Banach spaces replaces the original Hilbert spaces model.

Definition 2.1. *Let (X, d) be a metric space and B be a Banach space. A map $f : X \to B$ is said to be a* coarse embedding *of X into B if there exist two non-decreasing functions ρ_1 and ρ_2 from $[0, +\infty)$ to $(-\infty, +\infty)$ with $\lim_{r \to +\infty} \rho_i(r) = +\infty$ for $i = 1, 2$, such that, for all $x, y \in X$:*

$$\rho_1(d(x, y)) \leq \|f(x) - f(y)\| \leq \rho_2(d(x, y)).$$

Remark 2.2. *Every metric space (X,d) admits a coarse embedding into $\ell^\infty(X)$ via, for a fixed $x_0 \in X$, the map*

$$f : x \to \{y \mapsto d(x,y) - d(x_0,y)\}.$$

In fact, f is an isometric embedding.
Moreover, for a finitely generated group Γ endowed with the word metric d induced by a finite generating set, the same map $f : g \mapsto \{h \mapsto d(g,h) - d(e_\Gamma, h)\}$ is a proper cocycle with respect to the left regular representation on $\ell^\infty(\Gamma)$. Hence, every finitely generated group has property PL^∞.

Definition 2.3. *A metric space (X,d) is said to admit a* fibred coarse embedding into *a Banach space B, if there exist :*

1. *a field of Banach spaces $(B_x)_{x\in X}$ over X such that each B_x is affinely isometric to B ;*

2. *a section $s : X \to \bigsqcup_{x\in X} B_x$ (i.e. $s(x) \in B_x$) ;*

3. *two non-decreasing functions ρ_1 and ρ_2 from $[0,+\infty)$ to $(-\infty,+\infty)$ with $\lim_{r\to+\infty} \rho_i(r) = +\infty$ for $i = 1,2$ such that :*
   for any $r > 0$, there exists a bounded subset $K_r \subset X$ for which there exists a "trivialization"

   $$t_C : (B_x)_{x\in C} \to C \times B$$

   for each subset $C \subset X \smallsetminus K_r$ of diameter less than r ; that is, a map from $(B_x)_{x\in C}$ to the constant field $C \times B$ over B such that the restriction to the fibre B_x for $x \in C$ is an affine isometry $t_C : B_x \to B$, satisfying the following conditions :
   i) for any $x, y \in C$, $\rho_1(d(x,y)) \le \|t_C(x)(s(x)) - t_C(y)(s(y))\|_B \le \rho_2(d(x,y))$;

   ii) for any two subsets $C_1, C_2 \subset X \smallsetminus K_r$ of diameter less than r with $C_1 \cap C_2 \neq \emptyset$, there exists an affine isometry $t_{C_1C_2} : B \to B$ such that $t_{C_1}(x) \circ t_{C_2}(x)^{-1} = t_{C_1C_2}$, for all $x \in C_1 \cap C_2$.

Remark 2.4. *Let (X,d) be a metric space and B be a Banach space. If X coarsely embeds into B then X fibred coarsely embeds into B. In fact, if $f : X \to B$ is a coarse embedding with control functions ρ_1, ρ_2 then a fibred coarse embedding of X into B is given by :*

1. *the field of Banach spaces $(B_x)_{x\in X}$ where $B_x := B$ for all $x \in X$;*
2. *the section $s : x \mapsto f(x) \in B = B_x$;*
3. *the two control functions ρ_1 and ρ_2 and for each $r > 0$, considering $K_r = \emptyset$, for all C of diameter less than r, the "trivial" trivialisation given by, for $x \in X$, $t_C(x) = Id_B$ (which satisfies condition i) and ii) since f is a coarse embedding).*

The following proposition is proved by Chen, Wang and Wang in [CWW13] (see Proposition 1.4) in the general setting of fibred coarse embeddings into metric spaces.

Proposition 2.5. *Let Γ be a finitely generated, residually finite group. If Γ acts properly isometrically on a metric space Y, then any box space $\Box\Gamma$ admits a fibred coarse embedding into Y.*

We can then the reformulate this statement in the context of property $P\mathcal{B}$:

Corollary 2.6. *Let Γ be a finitely generated, residually finite group and $\mathcal{B}$ a class of Banach spaces. If Γ has property $P\mathcal{B}$, then any box space $\Box\Gamma$ admits a fibred coarse embedding into some Banach space $B \in \mathcal{B}$.*

3 Proof of the main results

Definition 3.1. *Let Γ be a finitely generated group and r be a non-negative real.*

i) Let X be a set. A map $\alpha : \Gamma \times X \to X$ is said to be a r-locally action of G on X *if :*
– for all $g \in \Gamma$ such that $d(e, g) < r$, $\alpha(g) : X \to X$ is a bijection ;
– for all $g, h \in \Gamma$ such that $d(e, g)$, $d(e, h)$, $d(e, gh)$ are less than r,

$$\alpha(gh) = \alpha(g)\alpha(h).$$

ii) Let B be a Banach space. A map $\pi : \Gamma \times B \to B$ is said to be a r-locally isometric representation of G on B *if π is a r-locally action of Γ on B and for all $g \in \Gamma$ such that $d(e, g) < r$, $\pi(g) : B \to B$ is a linear isometry.*

In this case, a map $b : \Gamma \to B$ such that, for all $g, h \in \Gamma$ such that $d(e, g)$, $d(e, h)$, $d(e, gh)$ are less than r, $\pi(g)b(h) + b(g) = b(gh)$, is called a r-locally cocycle with respect to π.

iii) Let B be a Banach space. A map $\alpha : G \times B \to B$ is called a r-locally isometric affine action of Γ on B *if it can be written as $\alpha(g)\cdot = \pi(g)\cdot + b(g)$ where π is a r-locally isometric representation and b is a r-locally cocycle with respect to π.*

Using the notion of ultrafilters and ultraproducts, one can build a global isometric affine action from a family of r-locally isometric affine actions with $r \to +\infty$.

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}^*$ i.e. $\mathcal{U}$ is a subset of $\mathcal{P}(\mathbb{N}^*)$ stable by intersection such that :
- the empty set $\emptyset$ does not belong to $\mathcal{U}$,
- for all $A, B \in \mathcal{P}(X)$ such that $A \subset B$, $A \in \mathcal{U}$ implies $B \in \mathcal{U}$,
- for all $A \in \mathcal{P}(X)$, $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.
- finite subsets of $\mathbb{N}^*$ do not belong to $\mathcal{U}$.

The $\mathcal{U}$-limit of bounded real valued sequence $(x_r)_{r\in\mathbb{N}^*}$ is the unique $x \in \mathbb{R}$ denoted by $\lim_{\mathcal{U}} x_r$ such that for all $\varepsilon > 0$, the set $\{r \in \mathbb{N}^* \mid |x_r - x| \leq \varepsilon\}$ belongs to $\mathcal{U}$.

Definition 3.2. *Let $(B_r)_{r\in\mathbb{N}^*}$ be a family of Banach spaces and consider the space $\ell^\infty(\mathbb{N}^*, (B_r)_{r\in\mathbb{N}^*})$ of sequences $(a_r)_{r\in\mathbb{N}^*}$ satisfying that there exists $K \geq 0$ such that for all $r \in \mathbb{N}^*$, $a_r \in B_r$ with $\|a_r\|_{B_i} \leq K$.*

The ultraproduct $B_{\mathcal{U}}$ of the family $(B_r)_{r\in\mathbb{N}^*}$ with respect to a non-principal ultrafilter $\mathcal{U}$ *is the closure of the space $\ell^\infty(\mathbb{N}^*, (B_r))/ \sim_{\mathcal{U}}$ endowed with the norm $\|(a_r)\|_{B_{\mathcal{U}}} := \lim_{\mathcal{U}} \|a_r\|_{B_r}$ where, for $(a_r), (b_r) \in \ell^\infty(\mathbb{N}^*, (B_r))$,*

$$(a_r) \sim_{\mathcal{U}} (b_r) \text{ if, and only if, } \|(a_r) - (b_r)\|_{B_{\mathcal{U}}} = 0.$$

Lemma 3.3. *Let Γ be a finitely generated group, $(B_r)_{r\in\mathbb{N}^*}$ be a family of Banach spaces and $B_{\mathcal{U}}$ be the ultraproduct of the family (B_r) with respect to a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}^*$. For each $r \in \mathbb{N}^*$, assume that Γ admits a r-locally isometric affine action α_r on B_r with $\alpha_r(g)\cdot = \pi_r(g)\cdot + b_r(g)$.*
If, for all $g \in \Gamma$, $(b_r(g))_{r\in\mathbb{N}^*}$ belongs to $B_{\mathcal{U}}$, then there exists an isometric affine action α of G on $B_{\mathcal{U}}$ of the family (B_r) such that $\alpha(g)\cdot = \pi(g)\cdot + b(g)$ where π is an isometric representation of Γ on $B_{\mathcal{U}}$ and $b : G \to B_{\mathcal{U}}$ is a cocycle with respect to π satisfying, for $g \in \Gamma$:

$$b(g) = (b_r(g))_{r\in\mathbb{N}^*}.$$

Proof. For $g \in \Gamma$, we define $\pi(g) : B_{\mathcal{U}} \to B_{\mathcal{U}}$ by, for $a = (a_r)_{r\in\mathbb{N}^*} \in B_{\mathcal{U}}$,

$$\pi(g)a = (\pi_r(g)a_r)_{r\in\mathbb{N}^*};$$

and we set $b(g) = (b_r(g))_{r\in\mathbb{N}^*} \in B_{\mathcal{U}}$.
Let $g, h \in \Gamma$. For all $r \in \mathbb{N}^*$ such that $r > \max(d(e, g), d(e, h), d(e, gh))$, we have, for all $(a_r) \in B_{\mathcal{U}}$, $\pi_r(g)\pi_r(h)a_r = \pi_r(gh)a_r$ and then the set $\{r \in \mathbb{N}^* \mid \pi_r(g)\pi_r(h)a_r = \pi_r(gh)a_r\}$ belongs to $\mathcal{U}$. Hence,

for all $g, h \in \Gamma$, $\pi(g)\pi(h) = \pi(gh)$. Now, for $g \in \Gamma$, since for all r large enough, $\pi_r(g)$ is an isometric isomorphism of B_r, it follows, by a similar argument, that $\pi(g)$ is an isometric isomorphism of $\mathcal{B}_{\mathcal{U}}$.
Thus, π is an isometric representation of Γ on $B_{\mathcal{U}}$.
Let $g, h \in \Gamma$. For all $r \in \mathbb{N}^*$ such that $r > \max(d(e, g), d(e, h), d(e, gh))$, we have $b_r(gh) = \pi_r(g)b_r(h) + b_r(g)$. Hence, for all $g, h \in \Gamma$, $b(gh) = \pi(g)b(h) + b(g)$ and then, b is a cocycle with respect to π. It follows that the map α such that $\alpha(g)\cdot = \pi(g)\cdot + b(g)$ is an isometric affine action of Γ on $B_{\mathcal{U}}$. ☐

Proof of Proposition 1.2. The case $p = \infty$ is trivial (see Remark 2.2).
Let $1 \leq p < \infty$. Let $\{\Gamma_n\}_{n \in \mathbb{N}^*}$ be a nested sequence of finite index normal subgroups of Γ with trivial intersection such that the associated box space $\square\Gamma$ admits a coarse embedding f into a L^p space denoted by B with control functions ρ_1, ρ_2.
Let $n \in \mathbb{N}^*$ and denote $X_n := \Gamma/\Gamma_n$. Let us consider the Banach space $\bigoplus_{z \in X_n} B$ endowed with the following norm : for a vector $\xi = \bigoplus_{z \in X_n} \xi_z$,

$$\|\xi\|_p = \left(\sum_{z \in X_n} \|\xi_z\|_B^p \right)^{\frac{1}{p}}.$$

For $x \in X_n$, we define the following vector of $\bigoplus_{z \in X_n} B$:

$$\tilde{b}_n(x) := \frac{1}{(\# X_n)^{\frac{1}{p}}} \bigoplus_{z \in X_n} (f(zx) - f(z));$$

and let $\tilde{\sigma}_n$ be the isometric representation of X_n on $\bigoplus_{X_n} B$ such that for $\xi = \bigoplus_{z \in X_n} \xi_z$,

$$\tilde{\sigma}_n(x)\xi = \bigoplus_{z \in X_n} \xi_{zx}.$$

Then $\tilde{b}_n : X_n \to \bigoplus_{X_n} B$ is a cocycle with respect to $\tilde{\sigma}_n$. In fact, since for $x, y, z \in X_n$, we have $f(zxy) - f(z) = (f(zxy) - f(zx)) + (f(zx) - f(z))$, it follows that :

$$\begin{aligned} \tilde{b}_n(xy) &= \tfrac{1}{(\# X_n)^{\frac{1}{p}}} \textstyle\bigoplus_{z \in X_n} (f(zxy) - f(zx)) + \tfrac{1}{(\# X_n)^{\frac{1}{p}}} \textstyle\bigoplus_{z \in X_n} (f(zx) - f(z)), \\ \tilde{b}_n(xy) &= \tilde{\sigma}_n(x)\tilde{b}_n(y) + \tilde{b}_n(x). \end{aligned}$$

Moreover, since f is a coarse embedding, we have, for all $x \in X_n$:

$$\rho_1(d_{X_n}(x, e)) \leq \|\tilde{b}_n(x)\|_p \leq \rho_2(d_{X_n}(x, e)),$$

where e is the identity element of X_n.

Now, for each $r \in \mathbb{N}^*$, choose n_r such that the canonical quotient map $\pi_{n_r} : \Gamma \twoheadrightarrow X_{n_r}$ is r-isometric and define $\sigma_r := \tilde{\sigma}_{n_r} \circ \pi_{n_r}$ and $b_r := \tilde{b}_{n_r} \circ \pi_{n_r}$. Thus, for every r, b_r is a cocycle with respect to the isometric representation σ_r of Γ on $\bigoplus_{X_{n_r}} B$ and we have, for $g \in \Gamma$ such that $d_\Gamma(g, e_\Gamma) < r$:

$$\rho_1(d_\Gamma(g, e)) \leq \|b_r(g)\|_p \leq \rho_2(d_\Gamma(g, e_\Gamma)). \quad (*)$$

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}^*$ and $B_{\mathcal{U}}$ be the ultraproduct of $\left(\bigoplus_{X_{n_r}} B \right)_{r \in \mathbb{N}^*}$. For each r, the map α_r defined by $\alpha_r(g)\cdot := \pi_r(g)\cdot + b_r(g)$ for $g \in \Gamma$, is an isometric affine action. By $(*)$, for all $g \in \Gamma$, $(b_r(g))_{r \in \mathbb{N}^*}$ belongs to $B_{\mathcal{U}}$.
Hence, by Lemma 3.3, there exists an isometric affine action α of Γ on $B_{\mathcal{U}}$ such that $b : g \mapsto (b_r(g))$ is a cocycle for this action. Moreover, for $g \in \Gamma$, since for all r large enough, $\rho_1(d_\Gamma(g, e)) \leq \|b_r(g)\|_p$, we have :

$$\rho_1(d_\Gamma(g, e)) \leq \|b(g)\|_{B_{\mathcal{U}}};$$

hence α is proper. As the class of L^p spaces is closed under p-normed powers and ultraproduct, it follows that Γ has property PL^p. ☐

For the next proposition, the steps of the proof are essentially the same as in the proof of Proposition 1.2. But, in this case, since for a given constant r, the trivialization of a fibred coarse embedding is defined on subsets of diameter less than r, we need to "r-localize" our construction of isometric affine actions of the quotient groups Γ/Γ_{n_r}.

Proposition 3.4. *Let $1 \leq p \leq \infty$ and let Γ be a finitely generated, residually finite group. If a box space $\square\Gamma$ of Γ admits a fibred coarse embedding into some L^p space, then Γ has property PL^p.*

Proof. The case $p = \infty$ is trivial (see Remark 2.2).
Let $1 \leq p < \infty$. Let $\{\Gamma_n\}_{n\in\mathbb{N}^*}$ be a nested sequence of finite index normal subgroups of Γ with trivial intersection such that the associated box space $\square\Gamma$ admits a fibred coarse embedding into a L^p space denoted by B.
We set $X_n = \Gamma/\Gamma_n$ and $X = \bigsqcup_{n\in\mathbb{N}^*} X_n (= \square\Gamma)$. Let $r \in \mathbb{N}^*$. By Definition 2.3, there exist K_r and a trivialization t_C for each $C \subset X \smallsetminus K_r$ of diameter less than r satisfying conditions $i)$ and $ii)$.
Now, choose n_r large enough such that $X_{n_r} \subset X \smallsetminus K_r$ and the quotient map $\pi_{n_r} : \Gamma \twoheadrightarrow X_{n_r}$ is r-isometric, i.e., for each subset $Y \subset \Gamma$ of diameter less than r, $(\pi_{n_r})_{|Y}$ is an isometry onto its image.
For $z \in X_{n_r}$, we denote by $C_z := \{x \in X_{n_r} \mid d_{X_{n_r}}(z,x) < r\}$ the r-ball centered in z of X_{n_r} and we set, for $x \in X_{n_r}$, the following vector $c_r^z(x)$ of B :

$$c_r^z(x) := \begin{cases} t_{C_z}(z)(s(z)) - t_{C_z}(zx)(s(zx)) & \text{if } d_{X_{n_r}}(e,x) < r \text{ (i.e } x \in C_e); \\ 0 & \text{otherwise,} \end{cases}$$

where e is the identity element of X_{n_r}. Notice that, by Definition 2.3 3. i) for any $z \in X_{n_r}$ and any $x \in C_e$, $\rho_1(d_{X_{n_r}}(e,x)) \leq \|c_r^z(x)\|_B \leq \rho_2(d_{X_{n_r}}(e,x))$.
Let us consider the map $\tilde{b}_r : X_{n_r} \to \bigoplus_{z\in X_{n_r}} B$, defined by, for $x \in X_{n_r}$:

$$\tilde{b}_r(x) = \frac{1}{(\# X_{n_r})^{\frac{1}{p}}} \bigoplus_{z\in X_{n_r}} c_r^z(x).$$

We endow $\bigoplus_{z\in X_{n_r}} B$ with the norm induced by the norm of ℓ_p^n i.e. for $\xi = \bigoplus_{z\in X_{n_r}} \xi_z$,

$$\|\xi\|_p = \left(\sum_{z\in X_{n_r}} \|\xi_z\|_B^p \right)^{\frac{1}{p}}.$$

Hence, for $x \notin C_e$, $\tilde{b}_r(x)$ vanishes, and for $x \in C_e$, $\rho_1(d_{X_{n_r}}(e,x)) \leq \|\tilde{b}_r(x))\|_p \leq \rho_2(d_{X_{n_r}}(e,x))$.

We claim that $\tilde{b}_r(x)$ is a r-locally cocycle for a r-locally isometric representation $\tilde{\sigma}_r$ that we define as follows :
For $x \in C_e$ and $z \in X_{n_r}$, let $\rho_{C_z C_{zx}}$ be the linear part of the affine isometry $t_{C_z C_{zx}} : B \to B$. We define $\tilde{\sigma}_r(x) : \bigoplus_{z\in X_{n_r}} B \to \bigoplus_{z\in X_{n_r}} B$ by, for $\xi = \bigoplus_{z\in X_{n_r}} \xi_z$:

$$\tilde{\sigma}_r(x)(\xi) := \begin{cases} \bigoplus_{z\in X_{n_r}} \rho_{C_z C_{zx}}(\xi_{zx}) & \text{if } x \in C_e; \\ \xi & \text{otherwise.} \end{cases}$$

The map $\tilde{\sigma}_r$ is indeed a r-locally isometric representation : it is clear that $\tilde{\sigma}_r(x)$ is a isometric isomorphism for all $x \in X_{n_r}$; moreover it follows from Definition 2.3 3. ii) that $t_{C_z C_{zy}} \circ t_{C_{zy} C_{zyx}} = t_{C_z C_{zyx}}$ for all $x, y \in C_e$ with $d_{X_{n_r}}(e, yx) < r$, and then, $\rho_{C_z C_{zy}} \circ \rho_{C_{zy} C_{zyx}} = \rho_{C_z C_{zyx}}$. Hence, $\tilde{\sigma}_r(yx) = \tilde{\sigma}_r(y)\tilde{\sigma}_r(x)$.

Now, we have, for $x, y \in C_e$ with $d_{X_{n_r}}(e, yx) < r$, $\tilde{\sigma}_r(y)(\tilde{b}_r(x)) + \tilde{b}_r(y) = \tilde{b}_r(yx)$. In fact, by noticing that for an affine isometry T with linear part ρ, $\rho(x - y) = Tx - Ty$, we have :

$$
\begin{aligned}
\rho_{C_zC_{zy}}(c_r^{zy}(x)) &= \rho_{C_zC_{zy}}\left(t_{C_{zy}}(zy)(s(zy)) - t_{C_{zy}}(zyx)(s(zyx))\right), \\
&= t_{C_zC_{zy}} \circ t_{C_{zy}}(zy)(s(zy)) - t_{C_zC_{zy}} \circ t_{C_{zy}}(zyx)(s(zyx)) + t_{C_z}(z)(s(z)) - t_{C_z}(zy)(s(zy)), \\
\rho_{C_zC_{zy}}(c_r^{zy}(x)) &= t_{C_z}(zy)(s(zy)) - t_{C_z}(zyx)(s(zyx))
\end{aligned}
$$

since $t_{C_zC_{zy}} \circ t_{C_{zy}}(zy) = t_{C_z}(zy)$ (by Definition 2.3 3. ii)) .
Thus,

$$\rho_{C_zC_{zy}}(c_r^{zy}(x)) + c_r^z(y) = t_{C_z}(z)(s(z)) - t_{C_z}(zyx)(s(zyx)) = c_r^z(yx).$$

It follow that :

$$\tilde{\sigma}_r(y)(\tilde{b}_r(x)) + \tilde{b}_r(y) = \frac{1}{(\# X_{n_r})^{\frac{1}{p}}} \bigoplus_{z \in X_{n_r}} \left(\rho_{C_zC_{zy}}(c_r^{zy}(x)) + c_r^z(y)\right) = \tilde{b}_r(yx)$$

which proves our claim.

Now, let $\sigma_r := \tilde{\sigma}_r \circ \pi_{n_r}$ and $b_r = \tilde{b}_r \circ \pi_{n_r}$ be the lifts of $\tilde{\sigma}_r$ and $\tilde{b}_r$ to the r-ball $\{g \in \Gamma \mid d_\Gamma(e_\Gamma, g) < r\}$ of Γ and define $\sigma_r = Id$, $b_r = 0$ outside the r-ball of Γ. Then σ_r is a r-locally isometric representation action of Γ on $\bigoplus_{X_{n_r}} B$, b_r is a r-locally cocycle with respect to σ_r. Then the map α_r such that $\alpha_r(g)\cdot := \sigma_r(g)\cdot + b_r(g)$ is a r-locally isometric affine action of Γ on $\bigoplus_{X_{n_r}} B$ and we have, for $g \in \Gamma$ with $d_\Gamma(e_\Gamma, g) < r$:

$$\rho_1(d_\Gamma(e_\Gamma, g)) \leq \|b_r(g)\|_p \leq \rho_2(d_\Gamma(e_\Gamma, g)).$$

From these local isometric affine actions, we build a global isometric affine action of Γ thanks to Lemma 3.3.
Let $\mathcal{U}$ be a non principal ultrafilter on $\mathbb{N}^*$, and let $B_\mathcal{U}$ be the ultraproduct of the family $\left(\bigoplus_{X_{n_r}} B\right)_{r \in \mathbb{N}^*}$ with respect to $\mathcal{U}$. For each $r \in \mathbb{N}^*$, α_r is a r-locally isometric affine action of Γ on $\bigoplus_{X_{n_r}} B$ and since, for any $g \in \Gamma$, $\|b_r(g)\|_p \leq \rho_2(d_\Gamma(e_\Gamma, g))$ for all $r \in \mathbb{N}^*$, $(b_r(g))_{r \in \mathbb{N}^*}$ belongs to $B_\mathcal{U}$. Hence, by Lemma 3.3, there exists an isometric affine action α of Γ on $B_\mathcal{U}$ such that $b : g \mapsto (b_r(g))$ is a cocycle with respect to the linear part of this action. Moreover, since for any $g \in \Gamma$, $\rho_1(d_\Gamma(e_\Gamma, g)) \leq \|b_r(g)\|_p$ for all r large enough, we have, for all $g \in \Gamma$: $\rho_1(d_\Gamma(e_\Gamma, g)) \leq \|b(g)\|_{B_\mathcal{U}}$, and thus, α is proper.
As the class of L^p spaces is closed under p-normed powers and ultraproduct, it follows that Γ has property PL^p. □

Proof of Theorem 1.1. It follows from Corollary 2.6 and Proposition 3.4. □

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