

THE HAAGERUP PROPERTY IS NOT INVARIANT UNDER QUASI-ISOMETRY

MATHIEU CARETTE

WITH AN APPENDIX BY SYLVAIN ARNT, THIBAUT PILLON AND ALAIN VALETTE

ABSTRACT. Using the work of Cornulier-Valette and Whyte, we show that neither the Haagerup property nor weak amenability is invariant under quasi-isometry of finitely generated groups.

A central topic in geometric and measured group theory is the study among finitely generated groups of invariants for quasi-isometry (QI) and measure equivalence (ME) respectively. Despite what might seem from the classical setting of lattices in semisimple Lie groups, quasi-isometry does not imply measure equivalence in general nor vice versa. In fact beyond the world of semisimple Lie groups, a comparison of a few basic invariants reveals strikingly little intersection (See [BHV08, CCJJV01, CH14, Fur11] and references therein for the basics on QI and ME invariants).

- QI invariants: amenability, number of ends, growth, hyperbolicity, finite presentability and Dehn function, asymptotic cones,...
- ME invariants: amenability, property (T), ratio of L^2 -Betti numbers, the Haagerup property (also called a-(T)-menability), weak amenability and the Cowling-Haagerup constant,...

With the notable exception of amenability, each property listed above as a QI invariant is moreover known not to be a ME invariant. In the other direction, it is known that neither property (T) nor ratio of L^2 -Betti numbers are QI invariants. It is therefore natural to ask whether the Haagerup property and weak amenability, which are both natural generalizations of amenability, are also QI invariants.

We show that these two properties are not QI invariants, settling an open problem raised in [CTV07, p. 774].

Theorem 1. *There exists two finitely generated groups Γ , Λ which are quasi-isometric such that Γ has the Haagerup property and is weakly amenable and Λ has neither of those properties.*

Our examples are fundamental groups of graphs of \mathbf{Z}^2 's. The quasi-isometric classification of such groups is addressed by Whyte [Why10]. Cornulier-Valette [CV] characterized which such groups have the Haagerup property and which are weakly amenable. It turns out both answers depend on a holonomy map discussed in Section 0.1. We also discuss the key role of locally compact groups in Section 0.2. In particular allowing unimodular compactly generated locally compact groups instead of finitely generated groups, we give examples as in Theorem 1 where both groups are of the form $\mathbf{R}^2 \rtimes F$ where F is a finitely generated free group (see Remark 7).

2010 *Mathematics Subject Classification.* 20F65, 20E08, 22D05, 22D10.

Key words and phrases. Haagerup property, a-(T)-menability, weak amenability, quasi-isometry, generalized Baumslag-Solitar groups, equivariant L^p compression.

M.C. is a Postdoctoral Researcher of the F.R.S.-FNRS (Belgium).

T.P. is supported by grant FN 200020-149261/1 of the swiss SNF.

In the appendix, Arnt, Pillon and Valette show that for $1 \leq p \leq 2$ the vanishing of equivariant L^p compression is not a quasi-isometry invariant.

0.1. Generalized Baumslag-Solitar groups. Fix $n \in \mathbf{N}$ an integer. We consider the class \mathcal{GBS}_n of groups Γ acting cocompactly on a locally finite tree T such that all (vertex and edge) stabilizers are isomorphic to \mathbf{Z}^n . Equivalently, a group Γ is in \mathcal{GBS}_n if it is the fundamental group of a finite graph of groups where all edge and vertex groups are isomorphic to \mathbf{Z}^n [Ser80].

Let Γ and T be as above. Fix a vertex $v \in T$ and an isomorphism $\Gamma_v \cong \mathbf{Z}^n$. The action of Γ on Γ_v by commensuration induces a homomorphism to the abstract commensurator $\text{hol} : \Gamma \rightarrow \text{Comm}(\Gamma_v) = \text{GL}_n(\mathbf{Q})$ which we call the **holonomy map** (this is also sometimes called the modular homomorphism, especially for the class \mathcal{GBS}_1 [Lev07]). Note that h is well-defined up to conjugation in $\text{GL}_n(\mathbf{Q})$ (with the different possibilities coming from a different choice of basepoint w and a different identification $\Gamma_w \cong \mathbf{Z}^n$). In fact, the commensurability class of Γ_v inside Γ does not depend on the chosen tree T (as soon as Γ is not amenable, equivalently Γ stabilizes no vertex, no line and no end of T) see e.g. the proof of Lemma 8.5 in [GL07]. From now on, we view the holonomy as a map to $\text{GL}_n(\mathbf{R})$.

We say that Γ, Γ' have **Hausdorff equivalent holonomy** if there is a compact subset $K \subset \text{GL}_n(\mathbf{R})$ and some $g \in \text{GL}_n(\mathbf{R})$ such that $\text{hol}(\Gamma) \subset g \text{hol}'(\Gamma') g^{-1} K$ and $g \text{hol}'(\Gamma') g^{-1} \subset \text{hol}(\Gamma) K$. In other words $\text{hol}(\Gamma)$ is at finite Hausdorff distance from some conjugate $\text{hol}'(\Gamma')$ for the word metric corresponding to some (equivalently any) compact generating set of $\text{GL}_n(\mathbf{R})$.

Whyte showed that the QI classification of \mathcal{GBS}_n groups is essentially governed by the holonomy map.

Theorem 2 ([Why10, Theorem 0.1]). *Among the class of groups in \mathcal{GBS}_n whose Bass-Serre tree T has infinitely many ends the following holds:*

- (1) *If two groups are quasi-isometric then they have Hausdorff equivalent holonomy.*
- (2) *Groups within a given Hausdorff equivalence class of holonomy divide into three quasi-isometry invariant subclasses:*
 - (a) *Those which are of the form $\mathbf{Z}^n \rtimes F$ for F a free subgroup of $\text{GL}_n(\mathbf{Z})$.*
 - (b) *Those which are virtually ascending HNN-extensions of some endomorphism $E : \mathbf{Z}^n \rightarrow \mathbf{Z}^n$. These are classified up to QI in [FM00].*
 - (c) *All groups not of the first two forms, all of which are in a single quasi-isometry class.*

Remark 3. Groups of the form (2b) are exactly those groups which are amenable. Groups of the form (2a) have a holonomy with discrete image in $\text{GL}_n(\mathbf{R})$.

Cornuier and Valette showed that both the Haagerup property and weak amenability of a \mathcal{GBS}_n group is determined by the image of its holonomy.

Theorem 4 ([CV, Theorem 1.6]). *Let $\Gamma \in \mathcal{GBS}_n$, with holonomy $\text{hol} : \Gamma \rightarrow \text{GL}_n(\mathbf{R})$. Then the following are equivalent:*

- (1) *Γ has the Haagerup property.*
- (2) *Γ is weakly amenable.*
- (3) *Γ has Cowling-Haagerup constant 1.*
- (4) *$\overline{\text{hol}(\Gamma)}$ is amenable.*

Proof of Theorem 1. Let $X = \mathbf{Z}^2 = \langle a, b \mid [a, b] \rangle$ and consider the subgroup $Y = \langle a, b^2 \rangle < X$. Consider furthermore the following matrices in $\text{SL}_2(\mathbf{Q})$:

$$H = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}; P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To show that Γ has the Haagerup property Cornulier and Valette realize Γ as a discrete subgroup in the locally compact group $\text{Aut}(T) \times G$ where T is the (6-regular) Bass-Serre tree of the graph of groups \mathbf{A} . They then proceed to show that G has the Haagerup property, and since so does $\text{Aut}(T)$, then Γ has the Haagerup property. On the other hand, the group Λ contains a finite index subgroup of $\mathbf{Z}^2 \rtimes \text{SL}_2(\mathbf{Z})$ which has property (T) relative to the subgroup \mathbf{Z}^2 , so that Λ cannot have the Haagerup property.

Concerning quasi-isometry, we note that in the forthcoming paper [CT] the author and Tessera use successive cocompact embeddings in (not necessarily unimodular) locally compact groups to show that Γ is quasi-isometric to G and Λ is quasi-isometric to L . Thus, our use of Whyte's result may be reduced to providing a quasi-isometry between the two semidirect products G and L . To do this, Whyte [Why10] produces a carefully chosen bilipschitz bijection $\eta : F_2 \rightarrow F_3$ such that there is a compact $K \subset \text{SL}_2(\mathbf{R})$ with the property that $\psi(g)\varphi(\eta(g))^{-1} \in K$ for each $g \in F_2$. This enables him to lift η to a fiber-preserving quasi-isometry $\tilde{\eta} : G \rightarrow L$. To produce the bijection, essential use is made of the fact that the maps ψ and φ are not proper.

Remark 7. The semidirect products G and L are arguably simpler (non-discrete) examples showing that the Haagerup property and weak amenability are not QI invariants among *unimodular* s.c. c.g. l.c. groups. Such examples cannot be obtained by cocompact inclusions $H < H'$ as this would force H and H' to be ME. Although it is an example of two quasi-isometric groups only one of which is Haagerup, the related cocompact inclusion $\mathbf{R}^2 \rtimes T_2 < \mathbf{R}^2 \rtimes \text{SL}_2(\mathbf{R})$ is uninteresting from the point of view of ME as $\mathbf{R}^2 \rtimes T_2$ is not unimodular (where $T_2 = \overline{\langle H, P \rangle}$ denotes the group described in equation (1)).

ACKNOWLEDGEMENTS

This note grew out of an ongoing collaboration with Romain Tessera, who I thank for helpful discussions. I am also grateful to Alain Valette for his comments on previous versions of this note.

APPENDIX A. EQUIVARIANT COMPRESSION IS NOT A QI-INVARIANT

BY S. ARNT, T. PILLON AND A. VALETTE

Equivariant compression was introduced as a way to quantify the Haagerup property. For G a finitely generated group, $p \in [1, +\infty[$, and $f : G \rightarrow L^p$ a G -equivariant map (with respect to some affine isometric action of G on L^p), the L^p -compression of f is:

$$\text{comp}_p(f) = \sup\{\alpha \in [0, 1] : \exists C > 0 : C|x^{-1}y|_S^\alpha - C \leq \|f(x) - f(y)\|_p \ \forall x, y \in G\},$$

where $|\cdot|_S$ denotes word length with respect to the finite generating set S in G ; clearly $\text{comp}_p(f)$ does not depend on the choice of S . Then the L^p -compression of G is $\alpha_p^\sharp(G) = \sup \text{comp}_p(f)$, where the supremum is taken on all G -equivariant maps $f : G \rightarrow L^p$. It is a result by Naor and Peres (see [NP11, Thm. 9.1]) that α_p^\sharp is a QI-invariant among finitely generated amenable groups. The purpose of this Appendix is to use Carette's examples from Theorem 1 above, to show that, as might be expected, α_p^\sharp is not QI-invariant among all finitely generated groups.

Theorem 8. *For $1 \leq p \leq 2$, the vanishing of α_p^\sharp is not a QI-invariant.*

Proof. We use the two groups Γ and Λ from the proof of Theorem 1. We will show that, for $1 \leq p \leq 2$, we have $\alpha_p^\sharp(\Gamma) = \frac{1}{p}$ while $\alpha_p^\sharp(\Lambda) = 0$.

- 1) $\alpha_p^\sharp(\Lambda) = 0$ for $p \in [1, 2]$. Indeed, assume by contradiction $\alpha_p^\sharp(\Lambda) > 0$ for some p . Then Λ admits a proper affine isometric action on L^p . By Corollary 6.23 of [CDH10] (using $1 \leq p \leq 2$), the group Λ has the Haagerup property, which is a contradiction.
- 2) For every $p \geq 1$, we have $\alpha_p^\sharp(\Gamma) = \max\{\frac{1}{p}, \frac{1}{2}\}$. This will follow from Theorem 6.3 in [CV], once we check the three assumptions. The first one is amenability of $\overline{\text{hol}(\Gamma)}$, which clearly holds. The second is that $\overline{\text{hol}(\Gamma)}$ should be co-compact in a closed, connected subgroup of $\text{GL}_2(\mathbf{R})$: here the upper triangular subgroup of $\text{SL}_2(\mathbf{R})$ does the job. The third one is that the inclusion $\mathbf{R}^2 \rtimes \text{hol}(\Gamma) \rightarrow \mathbf{R}^2 \rtimes \overline{\text{hol}(\Gamma)}$ should induce a quasi-isometry in restriction to \mathbf{R}^2 , where $\text{hol}(\Gamma)$ is endowed with the discrete topology in the first semi-direct product, and both semi-direct products are endowed with the word length associated to a compact generating subset. A sufficient condition for this to hold, is given by Proposition 2.5(b) of [CV]: it is enough that \mathbf{R}^2 is exponentially distorted in $\mathbf{R}^2 \rtimes \text{hol}(\Gamma)$, i.e. $\limsup_{m \rightarrow \infty} \frac{|mv|_S}{\log m} < \infty$ for every vector $v \in \mathbf{R}^2$ (where $|\cdot|_S$ is word length with respect to some compact generating set S). This holds because the matrix $H = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ lies in $\text{hol}(\Gamma)$. \square

We thank Mathieu Carette for offering us to write up our result as an Appendix to his paper.

REFERENCES

- [BHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette, *Kazhdan's property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008.
- [CCJJV01] Pierre-Alain Cherix, Michael Cowling, Paul Jolissaint, Pierre Julg, and Alain Valette, *Groups with the Haagerup property*, Progress in Mathematics, vol. 197, Birkhäuser Verlag, Basel, 2001.
- [CDH10] Indira Chatterji, Cornelia Druţu, and Frédéric Haglund, *Kazhdan and Haagerup properties from the median viewpoint*, Adv. Math. **225** (2010), no. 2, 882–921.
- [CT] Mathieu Carette and Romain Tessera, *Geometric rigidity and flexibility for groups acting on trees*, In preparation.
- [CH14] Yves de Cornulier and Pierre de la Harpe, *Metric geometry of locally compact groups*, Book in preparation, <http://arxiv.org/abs/1403.3796>, 2014.
- [CTV07] Yves de Cornulier, Romain Tessera, and Alain Valette, *Isometric group actions on Hilbert spaces: growth of cocycles*, Geom. Funct. Anal. **17** (2007), no. 3, 770–792.
- [CV] Yves de Cornulier and Alain Valette, *On equivariant embeddings of generalized Baumslag-Solitar groups*, To appear in Geom. Dedicata.
- [FM00] Benson Farb and Lee Mosher, *On the asymptotic geometry of abelian-by-cyclic groups*, Acta Math. **184** (2000), no. 2, 145–202.
- [Fur11] Alex Furman, *A survey of measured group theory*, Geometry, rigidity, and group actions, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 2011, pp. 296–374.
- [GL07] Vincent Guirardel and Gilbert Levitt, *Deformation spaces of trees*, Groups Geom. Dyn. **1** (2007), no. 2, 135–181.
- [KPV13] David Kyed, Henrik Densing Petersen, and Stefaan Vaes, *L^2 -Betti numbers of locally compact groups and their cross section equivalence relations*, Preprint, <http://arxiv.org/abs/1302.6753>, 2013.
- [Lev07] Gilbert Levitt, *On the automorphism group of generalized Baumslag-Solitar groups*, Geom. Topol. **11** (2007), 473–515.
- [NP11] Assaf Naor and Yuval Peres, *L_p compression, traveling salesmen, and stable walks*, Duke Math. J. **157** (2011), no. 1, 53–108.
- [Ser80] Jean-Pierre Serre, *Trees*, Springer-Verlag, Berlin, 1980, Translated from the French by John Stillwell.
- [Why10] Kevin Whyte, *Coarse bundles*, Preprint, <http://arxiv.org/abs/1006.3347>, 2010.

UNIVERSITÉ CATHOLIQUE DE LOUVAIN, IRMP, CHEMIN DU CYCLOTRON 2, BTE L7.01.01, 1348
LOUVAIN-LA-NEUVE, BELGIUM

E-mail address: `mathieu.carette@uclouvain.be`

UNIVERSITÉ D'ORLÉANS, UFR SCIENCES, BÂTIMENT DE MATHÉMATIQUES - RUE DE CHARTRES,
B.P. 6759 - 45067 ORLÉANS CEDEX 2, FRANCE

E-mail address: `sylvain.arnt@univ-orleans.fr`

INSTITUT DE MATHÉMATIQUES, UniMAIL, 11 RUE EMILE ARGAND, CH-2000 NEUCHÂTEL,
SWITZERLAND

E-mail address: `thibault.pillon@unine.ch`

E-mail address: `alain.valette@unine.ch`