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Large scale geometry and isometric affine actions on Banach spaces

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Résumé

Dans le premier chapitre, nous définissons la notion d'espaces à partitions pondérées qui généralise la structure d'espaces à murs mesurés et qui fournit un cadre géométrique à l'étude des actions isométriques affines sur des espaces de Banach pour les groupes localement compacts à base dénombrable. Dans un premier temps, nous caractérisons les actions isométriques affines propres sur des espaces de Banach en termes d'actions propres par automorphismes sur des espaces à partitions pondérées. Puis, nous nous intéressons aux structures de partitions pondérées naturelles pour les actions de certaines constructions de groupes : somme directe ; produit semi-directe ; produit en couronne et produit libre. Nous établissons ainsi des résultats de stabilité de la propriété PL^p par ces constructions. Notamment, nous généralisons un résultat de Cornulier, Stalder et Valette de la façon suivante : le produit en couronne d'un groupe ayant la propriété PL^p par un groupe ayant la propriété de Haagerup possède la propriété PL^p.

Dans le deuxième chapitre, nous nous intéressons aux espaces métriques quasi-médians - une généralisation des espaces hyperboliques à la Gromov et des espaces médians - et à leur propriétés. Après l'étude de quelques exemples, nous démontrons qu'un espace δ-médian est δ'-médian pour tout $\delta' \geq \delta$. Ce résultat nous permet par la suite d'établir la stabilité par produit directe et par produit libre d'espaces métriques - notion que nous développons par la même occasion.

Le troisième chapitre est consacré à la définition et l'étude d'une distance propre, invariante à gauche et qui engendre la topologie explicite sur les groupes localement compacts, compactement engendrés. Après avoir montré les propriétés précédentes, nous prouvons que cette distance est quasi-isométrique à la distance des mots sur le groupe et que la croissance du volume des boules est contrôlée exponentiellement.

Abstract

In the first chapter, we define the notion of spaces with labelled partitions which generalizes the structure of spaces with measured walls : it provides a geometric setting to study isometric affine actions on Banach spaces of second countable locally compact groups. First, we characterise isometric affine actions on Banach spaces in terms of proper actions by automorphisms on spaces with labelled partitions. Then, we focus on natural structures of labelled partitions for actions of some group constructions : direct sum ; semi-direct product ; wreath product and free product. We establish stability results for property PL^p by these constructions. Especially, we generalize a result of Cornulier, Stalder and Valette in the following way : the wreath product of a group having property PL^p by a Haagerup group has property PL^p.
In the second chapter, we focus on the notion of quasi-median metric spaces - a generalization of both Gromov hyperbolic spaces and median spaces - and its properties. After the study of some examples, we show that a δ-median space is δ'-median for all $\delta' \geq \delta$. This result gives us a way to establish the stability of the quasi-median property by direct product and by free product of metric spaces - notion that we develop at the same time.
The third chapter is devoted to the definition and the study of an explicit proper, left-invariant metric which generates the topology on locally compact, compactly generated groups. Having showed these properties, we prove that this metric is quasi-isometric to the word metric and that the volume growth of the balls is exponentially controlled.

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Introduction

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0.1 Motivations and context

A locally compact second countable group G has *Haagerup property* (or is *a-(T)-menable*) if there exists a proper continuous isometric affine action of G on a Hilbert space ; this property can be seen as a strong negation of Kazhdan's property (T) (an overview of the Haagerup property can be found in [CCJ$^+$01]). Groups having Haagerup property are known to satisfy the Baum-Connes conjecture by a result of Higson and Kasparov in [HK01] (see [Jul98] for further details). Haagerup property is closed by taking subgroups, direct products, amalgamated products over finite subsets but it is not stable by group extensions in general, even in the case of semi-direct products. However, Cornulier, Stalder and Valette recently proved in [CSV12] that it is stable by a particular kind of extension, namely the wreath product. They use for their proof the connexion between Haagerup property and spaces with measured walls, that we will now explain.

A *space with walls* is a pair (X, W) where X is a set and W is a family of partitions of X in two pieces called *walls* such that any pair of points of X is separated by finitely many walls. This notion was introduced by Haglund and Paulin in [HP98] and generalized in a topological setting by Cherix, Martin and Valette in [CMV04] to *space with measured walls* (see Definition 1.3.16). It was gradually realised that the Haagerup property is equivalent to the existence of a proper action on a space with measured walls ; more precisely, we have the following theorem : *a locally compact second countable group has the Haagerup property if, and only if, it acts properly by automorphisms on a space with measured walls.* Using results of Robertson and Steger (see [RS98]), Cherix, Martin and Valette in [CMV04], proved this theorem for discrete groups and Chatterji, Drutu and

Haglund extended the equivalence to locally compact second countable groups using the notion of median metric spaces in [CDH10]. In this context, we explore in Chapter 2 a generalization of the structure of median spaces namely, the δ-median spaces. The stability of the Haagerup property by wreath product was established in [CSV12] by constructing a space with measured walls from the structures of measured walls on each factor, and moreover, in the same article, Cornulier, Stalder and Valette generalized this result to the permutational wreath product (see Definition 1.5.1) when the index set I is a quotient by a co-Haagerup subgroup of the shifting group G (see [CI11] for a counter example when the pair (G, I) has relative property (T)). This result led to the first example of Haagerup groups which are not weakly amenable in the sense of [CH89].
The notion of Haagerup property naturally extends to proper isometric affine action on Banach spaces. Recent works have been made about isometric actions on Banach spaces : in [HP06], Haagerup and Przybyszewska showed that every locally compact second countable group G acts properly by affine isometries on the reflexive Banach space $\bigoplus^2_{n\in\mathbb{N}} L^{2n}(G,\mu)$ where μ is the Haar measure ; Cornulier, Tessera, Valette introduced in [CTV08] property (BP_0^V) for V a Banach space as a tool to show that the simple Lie group $G = Sp(n, 1)$ acts properly by isometries on $L^p(G)$ for $p > 4n + 2$; in [BFGM07], Bader, Furman Gelander and Monod, studied an analog of property (T) in terms of L^p spaces and more generally, of superreflexive Banach spaces. One of the motivation of this topic is given by a recent result of Kasparov and Yu in [KY12] which asserts that the existence of coarse embeddings of a finitely generated group in a uniformly convex Banach space implies the coarse geometric Novikov conjecture for this group. See [Now13] for an overview of results and questions about isometric affine actions on Banach spaces.
We will focus on specific Banach spaces, namely, L^p spaces. For $p \geq 1$, we say that a locally compact second countable group G has *property* PL^p (or is *a-*FL^p*-menable*) if there exists a proper continuous isometric affine action on a L^p space. See for instance [CDH10], for a caracterisation of property PL^p for $p \in [1, 2]$ in terms of Haagerup property. An important example is the following theorem due to Yu (see [Yu05]) : *let* Γ *be a discrete Gromov hyperbolic group. Then there exists* $p \geq 2$ *such that* Γ *has property* PL^p*.* Yu proved this result by giving an explicit proper isometric affine action of Γ on $\ell^p(\Gamma \times \Gamma \mid d(x, y) \leq R)$ using a construction of Mineyev in [Min01] ; see [Bou11] or [Nic12] for other proofs of this result in terms of boundaries of G. A remarkable consequence is that there exists infinite groups with property (T) (and hence, without Haagerup property) which have property PL^p for some $p > 2$.

In Chapter 1, we define a generalization of the structure of spaces with measured walls, namely, the structure of spaces with labelled partitions which provides a flexible framework, in terms of geometry and stability by various type of group constructions, for isometric affine actions on Banach spaces.

In this geometric context, the metric on the groups considered is a crucial data. For compactly generated groups, the "canonical" (for a fixed finite generating set) metric is the word metric but this metric need not generate the topology of the group.

Chapter 3 is devoted to answer the following question :

"Is there an explicit topological analog of the word metric on locally compact, compactly generated groups ?".

We already have a good overview about questions of metrizability of topological groups : Birkhoff in [Bir36] and Kakutani in [Kak36] showed independently that a topological group is metrizable if and only if it is Hausdorff and there exists a countable fundamental system of neighbourhoods of the identity element. In this case, the topology can be defined by a left(or right)-invariant metric (see for instance, [Die69]). But the metric built to prove this fact need not be a proper metric. As stated in [LMR00] in the case of compactly generated, second countable groups and in [HP06] in the case of locally compact, second countable groups, given a left-invariant metric which generates the topology, one can define a new metric which is plig (proper, left-invariant, generates the topology) ; futhermore, the previous result was already established in [Str74] ; in this paper, Struble showed the following theorem : a locally compact group G admits a plig metric if, and only if G is second countable.

0.2 Organisation of the text and statement of results

In Chapter 1, we define the notion of *spaces with labelled partitions* which generalizes the structure of space with measured walls in the general setting of continuous isometric affine actions on Banach spaces, and more particularly in the L^p case (see Section 1.3). We establish in Paragraph 1.3.3 the following result which links isometric affine actions on Banach spaces and actions by automorphisms on spaces with labelled partitions :

Theorem 1. *Let G be topological group.*

1. *If G acts (resp. acts properly) continuously by affine isometries on a Banach space B then there exists a structure $(G, \mathcal{P}, F(\mathcal{P}))$ of space with labelled partitions on G such*

that G acts (resp. acts properly) continuously by automorphisms on $(G, \mathcal{P}, F(\mathcal{P}))$ *via its left-action on itself. Moreover, there exists a linear isometric embedding* $F(\mathcal{P}) \hookrightarrow B$.

2. *If G acts (resp. acts properly) continuously by automorphisms on a space with labelled partitions* $(X, \mathcal{P}, F(\mathcal{P}))$ *then there exists a (resp. proper) continuous isometric affine action of G on a Banach space B. Moreover, B is a closed subspace of* $F(\mathcal{P})$.

This theorem can be rephrased in the particular case of L^p spaces as follows :

Corollary 2. *Let* $p \geq 1$ *with* $p \notin 2\mathbb{Z} \smallsetminus \{2\}$ *and G be a topological group. G has property* PL^p *if, and only if, G acts properly continuously by automorphisms on a space with labelled partitions* $(X, \mathcal{P}, F(\mathcal{P}))$ *where* $F(\mathcal{P})$ *is isometrically isomorph to a closed subspace of an* L^p *space.*

In Section 1.4, we give a natural construction of a space with labelled partitions on direct sums and we exhibit an explicit proper action by automorphisms on this space given proper actions by automorphisms on each factor of the direct sum.
In Paragraph 1.4.3, given groups G_1, G_2 acting properly by automorphisms on spaces with labelled partitions, we observe that if a morphism $\rho : G_2 \to \mathrm{Aut}(G_1)$ "preserves" the structure of labelled partitions of G_1, then the semi-direct product $G_1 \rtimes_\rho G_2$ acts properly by automorphisms on the natural space with labelled partitions of the direct product; more precisely, we prove :

Theorem 3. *Let* $(X_1, \mathcal{P}_1, F_1(\mathcal{P}_1))$,$(X_2, \mathcal{P}_2, F_2(\mathcal{P}_2))$ *be spaces with labelled partitions and* G_1, G_2 *be topological groups acting continuously by automorphisms on, respectively,* $(X_1, \mathcal{P}_1, F_1(\mathcal{P}_1))$ *and* $(X_2, \mathcal{P}_2, F_2(\mathcal{P}_2))$ *via* τ_1 *and* τ_2.
Let $\rho : G_2 \to Aut(G_1)$ *be a morphism of groups such that* $(g_1, g_2) \mapsto \rho(g_2)g_1$ *is continuous for the product topology on* $G_1 \times G_2$.
Assume that there exists a continuous action by automorphisms of $G_1 \rtimes_\rho G_2$ *on* X_1 *which extends the* G_1 *action.*

Then the semi-direct product $G_1 \rtimes_\rho G_2$ *acts continuously by automorphisms on the natural structure of labelled partitions* $(X_1 \times X_2, \mathcal{P}, F_q(\mathcal{P}))$ *on the direct product of* $X_1 \times X_2$.
Moreover, if, for $i = 1, 2$, G_i *acts properly on* $(X_i, \mathcal{P}_i, F_i(\mathcal{P}_i))$, *then* $G_1 \rtimes_\rho G_2$ *acts properly on* $(X_1 \times X_2, \mathcal{P}, F(\mathcal{P}))$.

We apply these results in Section 1.5 to prove that the wreath product of a group with property PL^p by a group with the Haagerup property has property PL^p :

Theorem 4. *Let H, G be countable discrete groups, L be a subgroup of G and $p > 1$, with $p \notin 2\mathbb{Z} \smallsetminus \{2\}$. We denote by I the quotient G/L and $W = \bigoplus_I H$. Assume that G is Haagerup, L is co-Haagerup in G and H has property PL^p.*
Then the permutational wreath product $H \wr_I G = W \rtimes G$ has property PL^p.

The proof of this theorem combines the technics mentionned previously to build structures of spaces with labelled partitions on the direct sum and on the semi-direct product, and a construction of space with measured walls provided in [CSV12].

In Section 1.6, we define the notion of free product of spaces with labelled partitions and we build a natural structure of labelled partitions on the free product such that given group actions by automorphisms on each factor, the free product of groups acts by automorphisms on the natural space with labelled partitions on the free product. More precisely, we prove :

Theorem 5.
Let $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ and $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ be non-empty countable spaces with labelled partitions and $x_0 \in X$, $y_0 \in Y$ be basepoints. Let G and H be discrete countable groups acting (resp. acting properly) by automorphisms on $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ and $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ respectively such that no element of G fixes x_0 and no element of H fixes y_0.

Let $q \geq 1$. Then there exists a structure of space with labelled partitions $(M, \mathcal{P}_M, F(\mathcal{P}_M))$ on which $G * H$ acts (resp. acts properly) by automorphisms.

More precisely, $(M, \mathcal{P}_M, F(\mathcal{P}_M))$ is the natural space with labelled partitions on the direct product $M = (X * Y) \times (G * H)$ where :

- *on $X*Y$, we consider the natural space with labelled partitions $(X*Y, \mathcal{P}_{X*Y}, F_q(\mathcal{P}_{X*Y}))$ on the free product of $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ and $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$;*
- *on $G*H$, we consider the natural space with labelled partitions $(G*H, \Delta_{G*H}, F_q(\Delta_{G*H}))$ on the free product of the q-naive spaces with labelled partitions $(G, \Delta_G, \ell^q(\Delta_G))$, $(H, \Delta_H, \ell^q(\Delta_H))$ on, respectively, G and H.*

In terms of property PL^p, the previous theorem implies the following stability result :

Corollary 6. *Let $p \geq 1$ with $p \notin 2\mathbb{Z} \smallsetminus \{2\}$ and G, H be discrete countable groups. G and H have property PL^p if, and only if, $G * H$ has property PL^p.*

In Chapter 2, we investigate a generalization of median spaces, namely, the quasi-median spaces. After the definition and some basic properties of quasi-median spaces, we prove, in Section 2.5 the following result by generalizing technics used in [Rol98] and [CDH10] for median spaces :

Theorem 7. *Let (X, d) be a δ-median space. Then (X, d) is δ'-median for all $\delta' \geq \delta$.*

This Theorem allows us to explore stability of the quasi-median property by constructions. In Section 2.6, we first state in Proposition 2.6.1 the stability by direct product and in a second part, we explore deeper the notion of free product of metric spaces initiated in Section 1.6, and we show the following :

Theorem 8. *A free product of quasi-median spaces is quasi-median for the free product metric.*

In Chapter 3, we define, on compactly generated topological groups, an explicit proper, left-invariant metric ρ_V which generates the topology. We give an extended study of ρ_V in order to show that this metric is plig and we prove that this metric is quasi-isometric to the word metric. We state the following result :

Theorem 9. *Let G be a locally compact, compactly generated group and V be a compact, symmetric, mobile and generating neighbourhood of the identity. Then ρ_V is a* **plig** *metric on G for which the balls have exponentially controlled growth.*
Moreover, if V' be a compact, symmetric, mobile and generating neighbourhood of the identity, then (G, ρ_V) and $(G, \rho_{V'})$ are **quasi-isometric***.*

Finally, in Section 3.5, we prove the following result :

Theorem 10. *Let V be a csg mobile neighbourhood of e. For every $x \in G$, there exists an optimal V-path from e to x.*

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Chapter 1

Spaces with labelled partitions

1.1 Introduction

In this chapter, we define and explore the structure of spaces with labelled partitions. This notion is meant to provide an analog of spaces with measured walls for isometric affine actions on Banach spaces. A crucial factor in the definition of space with labelled partition is the "geometric" understanding of the construction of Mineyev in [Min01] used by Yu in [Yu05] to exhibit a proper action of discrete hyperbolic groups on some ℓ^p space. Moreover, another inspiration for this definition comes from [CCJ$^+$01] Proposition 7.4.1 where Valette states the following geometric characterisation of the Haagerup property for locally compact groups : G has the Haagerup property if, and only if, there exists a metric space (X, d) on which G acts isometrically and metrically properly, a unitary representation π of G on a Hilbert space $\mathcal{H}_\pi$, and a continous map $c : X \times X \to \mathcal{H}_\pi$ such that :

1. Chasles' relation :
$$\text{for all } x, y, z \in X,\ c(x, z) = c(x, y) + c(y, z)\,;$$
2. G-equivariance condition :
$$\text{for all } x, y \in X,\ g \in G,\ c(gx, gy) = \pi(g)c(x, y)\,;$$
3. Properness condition :
$$\text{if } d(x, y) \to +\infty, \text{ then } \|c(x, y)\|_{\mathcal{H}_\pi} \to +\infty.$$

To emphasize the connection with this result, we use the same notation c (for $c$ocycle) for the separation map $c : X \times X \to F(\mathcal{P})$ associated with a set of labelling functions $\mathcal{P}$ (see Definition 1.3.3). In fact, an immediate consequence of Theorem 1 Statement 1. is that the separation map c_α of the set of labelled partitions associated with a proper isometric affine action α on a Banach space (see Definition 1.3.31), satisfies the conditions 1., 2., and 3. mentionned above.

We describe, in Part 2., the maps that preserve the structure of spaces with labelled partitions in order to define actions by automorphisms on a space with labelled partitions. This notion of homomorphisms of spaces with labelled partitions generalizes the notion of homomorphisms of spaces with measured walls (see [CDH10] Definition 3.5).
We discuss some constructions of spaces with labelled partitions for the direct sum, semi-direct product, wreath product and free product in Sections 1.4, 1.5 and 1.6 and we apply these constructions in the case of groups with property PL^p.

Subsequently, all topological groups we consider are assumed to be Hausdorff.

1.2 Preliminaries

1.2.1 Metrically proper actions

A pseudo-metric d on a set X is a symmetric map $d : X \times X \to \mathbb{R}_+$ which satisfies the triangle inequality and $d(x, x) = 0$. But unlike a metric, a pseudo-metric need not separate points.

Definition 1.2.1. *Let G be a topological group acting continuously isometrically on a pseudo-metric space (X, d_X). The G-action on X is said* metrically proper *if, for all (or equivalently, for some) $x_0 \in X$,*

$$\lim_{g \to \infty} d_X(g.x_0, x_0) = +\infty.$$

Let X be a set endowed with a pseudo-metric d. We put on X the following equivalence relation : for $x, x' \in X$, $x \sim x'$ if, and only if, $d(x, x') = 0$, and we denote by Y the quotient set $X/\sim$. Then we can define a metric $\tilde{d}$ on Y by setting, for $x, x' \in X$, $\tilde{d}([x], [x']) = d(x, x')$. Moreover, an isometric group action (X, d) preserves the classes of $\sim$ and then induces an isometric action on $(Y, \tilde{d})$.

Lemma 1.2.2. *Let G be a topological group acting continuously isometrically on a pseudo-metric space (X, d). The G-action on X is* metrically proper *if, and only if, the induced G-action on the quotient metric space $(Y, \tilde{d})$ is metrically proper.*

1.2.2 Isometric affine actions

Definition 1.2.3. *We say that the action of a topological group G on a topological space X is* strongly continuous *if, for all $x \in X$, the orbit map from G to X, $g \mapsto gx$ is continuous.*

Let G be a topological group and let $(B, \|.\|)$ be a Banach space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 1.2.4. *A* continuous isometric affine *action α of G on B is a strongly continuous morphism*

$$\alpha : G \longrightarrow \mathit{Isom}(B) \cap \mathit{Aff}(B).$$

Notice that if B is a real Banach space, then, by Mazur-Ulam Theorem,

$$\mathrm{Isom}(B) \cap \mathrm{Aff}(B) = \mathrm{Isom}(B).$$

Proposition 1.2.5. *A continuous isometric affine action α of G on B is characterised by a pair (π, b) where :*

— *π is a strongly continuous isometric representation of G on B,*

— *$b : G \to B$ is a continuous map satisfying the 1-cocycle relation : for $g, h \in G$,*

$$b(gh) = \pi(g)b(h) + b(g).$$

And we have, for $g \in G$, $x \in B$:

$$\alpha(g)x = \pi(g)x + b(g).$$

Definition 1.2.6. *Let α be a continuous isometric affine action of G on B. We say that α is* proper *if the action of G on the metric space $(B, d_{\|.\|})$ is metrically proper where $d_{\|.\|}$ is the canonical metric on B induced by the norm $\|.\|$.*

Proposition 1.2.7. *A continuous isometric affine action α of G on B is proper if, and only if*

$$\|b(g)\| \underset{g \to \infty}{\longrightarrow} +\infty.$$

Definition 1.2.8. *Let $p \geq 1$. We say that G has property PL^p (or is* a-FL^p-menable*) if there exists a proper continuous isometric affine action of G on a L^p space.*

1.2.3 On isometries of L^p-spaces

In general, for $p \geq 1$, a closed subspace of a L^p-space is not a L^p-space (exempt the special case $p = 2$) ; but, in [HJ81], Hardin showed the following result about extension of linear isometries on closed subspace of a L^p (here, we give a reformulation of this result coming from [BFGM07], Corollary 2.20) :

Theorem 1.2.9. *Let $p > 1$ with $p \notin 2\mathbb{Z} \setminus \{2\}$ and F be a closed subspace of $L^p(X, \mu)$. Let π be a linear isometric representation of a group G on F. Then there is a linear isometric representation α' of G on some other space $L^p(X', \mu')$ and a linear G-equivariant isometric embedding $F \hookrightarrow L^p(X', \mu')$.*

An immediate consequence is the following :

Corollary 1.2.10. *Let $p > 1$ with $p \notin 2\mathbb{Z} \setminus \{2\}$, F be a closed subspace of a L^p-space and G be a topological group. If G acts properly by affine isometries on F, then G has property PL^p.*

In Section 1.4, we embed linearly isometrically into L^p spaces some normed vector spaces isometrically isomorph to a direct sums of L^p spaces thanks to the following basic result :

Definition 1.2.11. *Let I be a countable index set, $(B_i, \|.\|_{B_i})_{i \in I}$ be a family of Banach spaces and $p \geq 1$. We call* ℓ^p-direct sum *of the family (B_i) the space :*

$$B = \bigoplus_{i \in I}^p B_i := \left\{ (x_i)_{i \in I} \in \prod_{i \in I} B_i \mid \sum_{i \in I} \|x_i\|_{B_i}^p < +\infty \right\},$$

and we denote, for $x = (x_i) \in B$,

$$\|x\|_p := \left(\sum_{i \in I} \|x_i\|_{B_i}^p \right)^{\frac{1}{p}}.$$

The space $B = \bigoplus_{i \in I}^p B_i$ endowed with the norm $\|.\|_p$ is a Banach space, and moreover, we have :

Proposition 1.2.12. *Let I be a countable index set, $p \geq 1$ and $(L^p(X_i, \mu_i))_{i \in I}$ be a family of L^p-spaces. Then $\left(\bigoplus_{i \in I}^p L^p(X_i, \mu_i), \|.\|_p \right)$ is isometrically isomorph to a L^p-space.*

1.3 Spaces with labelled partitions and actions on Banach Spaces

In this section we will introduce the structure of *space with labelled partitions* and record for further use a few basic properties.

1.3.1 Spaces with labelled partitions

1. Definitions

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Consider a set X and a function $p : X \to \mathbb{K}$. There is a natural partition $P = P(p)$ of X associated with p :
We have the following equivalence relation $\sim_p$ on X : for $x, y \in X$,

$$x \sim_p y \text{ if, and only if, } p(x) = p(y).$$

We define the partition associated with p by $P(p) = \{\pi_p^{-1}(h) \mid h \in X/\sim_p\}$ where π_p is the canonical projection from X to $X/\sim_p$.

Definition 1.3.1. *Let X be a set, and $\mathcal{P} = \{p : X \to \mathbb{K}\}$ be a family of functions.*

— *We say that p is a* labelling function *on X and the pair (P, p) is called a* labelled partition *of X.*

— *We say that $x, y \in X$ are* separated *by $p \in \mathcal{P}$ if $p(x) \neq p(y)$ and we denote by $\mathcal{P}(x|y)$ the set of all labelling functions separating x and y.*

Remark 1.3.2. *The terminology "x and y are separated by p" comes from the fact that, if we denote by P the partition of X associated with p, x and y are separated by p if, and only if, x and y belongs to two different sets of the partition P i.e. P separates x and y.*

Consider a set $\mathcal{P}$ of labelling functions on X, and the $\mathbb{K}$-vector space $\mathcal{F}(\mathcal{P}, \mathbb{K})$ of all functions from $\mathcal{P}$ to $\mathbb{K}$. Then we have a natural map $c : X \times X \to \mathcal{F}(\mathcal{P}, \mathbb{K})$ given by : for $x, y \in X$ and $p \in \mathcal{P}$,

$$c(x, y)(p) = p(x) - p(y).$$

Notice that p belongs to $\mathcal{P}(x|y)$ if, and only if, $c(x, y)(p) \neq 0$.

Definition 1.3.3. *Let X be a set and $\mathcal{P}$ be a family of labelling functions. The map $c : X \times X \to \mathcal{F}(\mathcal{P}, \mathbb{K})$ such that, for $x, y \in X$ and for $p \in \mathcal{P}$, $c(x,y)(p) = p(x) - p(y)$ is called the* separation map *of X relative to P.*

We now define the notion of space with labelled partitions :

Definition 1.3.4 (Space with labelled partitions)**.**
Let X be a set, $\mathcal{P}$ be a family of labelling functions from X to $\mathbb{K}$ and $(\mathcal{F}(\mathcal{P}), \|.\|)$ be a semi-normed space of $\mathbb{K}$-valued functions on $\mathcal{P}$ such that the quotient vector space $F(\mathcal{P})$ of $\mathcal{F}(\mathcal{P})$ by its subspace $\mathcal{F}(\mathcal{P})_0 = \{\xi \in \mathcal{F}(\mathcal{P}) \mid \|\xi\| = 0\}$ is a Banach space.

We say that $(X, \mathcal{P}, F(\mathcal{P}))$ is a space with labelled partitions if, for all $x, y \in X$:

$$c(x,y) : \mathcal{P} \to \mathbb{K} \text{ belongs to } \mathcal{F}(\mathcal{P}).$$

Definition 1.3.5. *If $(X, \mathcal{P}, F(\mathcal{P}))$ is a space with labelled partitions, we can endow X with the following pseudo-metric : $d(x,y) = \|c(x,y)\|$ for $x, y \in X$.*
We call d the labelled partitions pseudo-metric *on X.*

Remark 1.3.6. *If $(X, \mathcal{P}, F(\mathcal{P}))$ is a space with labelled partitions, then the separation map $c : X \times X \to F(\mathcal{P})$ is continuous where $X \times X$ is endowed with the product topology induced by the topology of (X, d).*

2. Actions on spaces with labelled partitions

Here, we describe the maps that preserve the structure of space with labelled partitions.

Definition 1.3.7 (homomorphism of spaces with labelled partitions)**.** *Let $(X, \mathcal{P}, F(\mathcal{P}))$, $(X', \mathcal{P}', F'(\mathcal{P}'))$ be spaces with labelled partitions and let $f : X \to X'$ be a map from X to X'.*

We say that f is a homomorphism of spaces with labelled partitions *if :*

1. *for any $p' \in \mathcal{P}'$, $\Phi_f(p') := p' \circ f$ belongs to $\mathcal{P}$,*
2. *for all $\xi \in F(\mathcal{P})$, $\xi \circ \Phi_f$ belongs to $F'(\mathcal{P}')$ and,*

$$\|\xi \circ \Phi_f\|_{F'(\mathcal{P}')} = \|\xi\|_{F(\mathcal{P})}.$$

An automorphism *of the space with labelled partitions* $(X, \mathcal{P}, F(\mathcal{P}))$ *is a bijective map* $f : X \to X$ *such that* f *and* f^{-1} *are homomorphisms of spaces with labelled partitions from* $(X, \mathcal{P}, F(\mathcal{P}))$ *to* $(X, \mathcal{P}, F(\mathcal{P}))$.

Remark 1.3.8.
- If f *is a homomorphism of spaces with labelled partitions, then* f *is an isometry from* X *to* X' *endowed with their respective labelled partitions pseudo-metrics ; indeed, for* $x, y \in X$,

$$d_X(x,y) = \|c(x,y)\|_{F(\mathcal{P})} = \|c(x,y) \circ \Phi_f\|_{F'(\mathcal{P}')} = \|c'(f(x), f(y))\|_{F'(\mathcal{P}')} = d_{X'}(f(x), f(y)),$$

since we have $c(x,y) \circ \Phi_f = c'(f(x), f(y))$.
- If f *is an automorphism of space with labelled partitions, the map* Φ_f *is a bijection :* $(\Phi_f)^{-1} = \Phi_{f^{-1}}$.

Proposition 1.3.9. *Let* $(X, \mathcal{P}, F(\mathcal{P}))$, $(X', \mathcal{P}', F'(\mathcal{P}'))$,$(X'', \mathcal{P}'', F''(\mathcal{P}''))$ *be spaces with labelled partitions and* $f : X \to X'$, $f' : X' \to X''$ *be homomorphisms of spaces with labelled partitions.*
We denote Φ_f *the map such that* $\Phi_f(p') := p' \circ f$, *for* $p' \in \mathcal{P}'$, *and* $\Phi_{f'}$ *the map such that* $\Phi_{f'}(p'') := p'' \circ f'$, *for* $p'' \in \mathcal{P}''$.
Then $f' \circ f$ *is a homomorphism of spaces with labelled partitions from* $(X, \mathcal{P}, F(\mathcal{P}))$ *to* $(X'', \mathcal{P}'', F''(\mathcal{P}''))$ *and we have, by denoting* $\Phi_{f' \circ f}(p'') := p'' \circ (f' \circ f)$ *:*

$$\Phi_f \circ \Phi_{f'} = \Phi_{f' \circ f}.$$

Proof. For all $p'' \in \mathcal{P}''$, we have :

$$\begin{aligned} \Phi_{f' \circ f}(p'') &= p'' \circ (f' \circ f) \\ &= (p'' \circ f') \circ f \\ &= \Phi_{f'}(p'') \circ f \text{ with } \Phi_{f'}(p'') \in \mathcal{P}' \text{ by Definition 1.3.7} \\ &= \Phi_f(\Phi_{f'}(p'')) \text{ and hence,} \\ \Phi_{f' \circ f}(p'') &= \Phi_f \circ \Phi_{f'}(p'') \in \mathcal{P} \text{ by Definition 1.3.7.} \end{aligned}$$

It follows that $\Phi_f \circ \Phi_{f'} = \Phi_{f' \circ f}$.

Now, let $\xi \in F(\mathcal{P})$. Since $\xi \circ \Phi_f$ belongs to $F'(\mathcal{P}')$,

$$\xi \circ \Phi_{f' \circ f} = (\xi \circ \Phi_f) \circ \Phi_{f'} \in F''(\mathcal{P}''),$$

and we clearly have, using the previous equality,

$$\|\xi \circ \Phi_{f' \circ f}\|_{F''(\mathcal{P}'')} = \|\xi \circ \Phi_f\|_{F'(\mathcal{P}')} = \|\xi\|_{F(\mathcal{P})}.$$

□

Remark 1.3.10. *Assume a group G acts by automorphisms on $(X, \mathcal{P}, F(\mathcal{P}))$. For $g \in G$, we denote by $\tau(g) : X \to X$, the map $x \mapsto \tau(g)x = gx$. Then, by Proposition 1.3.9, we have :*

$$\Phi_{\tau(g_2)} \circ \Phi_{\tau(g_1)} = \Phi_{\tau(g_1 g_2)}.$$

Definition 1.3.11. *Let $(X, \mathcal{P}, F(\mathcal{P}))$ be a space with labelled partitions and G be a topological group acting by automorphisms on $(X, \mathcal{P}, F(\mathcal{P}))$.*

- *We say that G acts* continuously *on $(X, \mathcal{P}, F(\mathcal{P}))$, if the G-action on (X, d) is strongly continuous.*

- *We say that G acts* properly *on $(X, \mathcal{P}, F(\mathcal{P}))$, if the G-action on (X, d) is metrically proper where d is the labelled partitions pseudo-metric on X.*

Remark 1.3.12. *Notice that if a topological Hausdorff group G acts properly continuously by automorphisms on a space $(X, \mathcal{P}, F(\mathcal{P}))$ with labelled partitions, then it is locally compact and σ-compact : in fact, let $x_0 \in X$; for $r > 0$, $V_r = \{g \in G \mid d(gx_0, x_0) \leq r\}$ is a compact neighbourhood of the identity element e in G since the action on (X, d) is strongly continuous and proper, and we have $G = \cup_{n \in \mathbb{N}^*} V_n$.*

Proposition 1.3.13. *Let G be a topological group. Assume G acts continuously by automorphisms on $(X, \mathcal{P}, F(\mathcal{P}))$.*
The G-action on $(X, \mathcal{P}, F(\mathcal{P}))$ is proper if, and only if, for every (resp. for some) $x_0 \in X$, $\|c(gx_0, x_0)\| \to \infty$ when $g \to \infty$.

Proof. It follows immediatly from the definition of a metrically proper action. □

Lemma 1.3.14 (pull back of space with labelled partitions)**.** *Let $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ be a space with labelled partitions, Y be a set and $f : Y \to X$ be a map. Then there exists a* pull back *structure of space with labelled partitions $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ turning f into a homomorphism.*
Moreover, if G acts on Y and G acts continuously by automorphisms on $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ such that f is G-equivariant, then G acts continuously by automorphisms on $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$.

Proof. We consider the family of labelling functions on Y :

$$\mathcal{P}_Y = \{p \circ f \mid p \in \mathcal{P}_X\},$$

and let c_Y be the separation map on Y associated with $\mathcal{P}_Y$.
Let $T : \mathrm{Vect}\big(c_Y(y, y') \mid y, y' \in Y\big) \to F_X(\mathcal{P}_X)$ be the linear map such that $T(c_Y(y, y')) = c_X(f(y), f(y'))$. The map T is well defined and is injective since, for every $p \in \mathcal{P}_X$,

$$c_X(f(y), f(y'))(p) = p \circ f(y) - p \circ f(y') = c_Y(y, y')(p \circ f).$$

On $\mathrm{Vect}\big(c_Y(y, y') \mid y, y' \in Y\big)$, we consider the following norm :
for $\xi \in \mathrm{Vect}\big(c_Y(y, y') \mid y, y' \in Y\big)$, we set,

$$\|\xi\|_{\mathcal{P}_Y} = \|T(\xi)\|_{F_X(\mathcal{P}_X)}.$$

And we set $F_Y(\mathcal{P}_Y) = \overline{\mathrm{Vect}\big(c_Y(y, y') \mid y, y' \in Y\big)}^{\|\cdot\|_{\mathcal{P}_Y}}$. Hence, by construction, $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ is a space with labelled partitions and f is clearly an homorphism from $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ to $(X, \mathcal{P}_X, F(\mathcal{P}_X))$ since, for all $y, y' \in Y$,

$$c_Y(y, y') \circ \Phi_f = c_X(f(y), f(y')),$$

where $\Phi_f(p) = p \circ f$ for $p \in \mathcal{P}_X$.

Assume that G acts on Y via τ_Y and G acts continuously by automorphisms on $(X, \mathcal{P}, F(\mathcal{P}))$ via τ_X, and f is G-equivariant. We denote, for $p \in \mathcal{P}_X$ and $g \in G$:
- $\Phi_{\tau_X(g)}(p) := p \circ \tau_X(g)$ and,
- $\Phi_{\tau_Y(g)}(p \circ f) := (p \circ f) \circ \tau_Y(g)$.
Since f is G-equivariant and $\mathcal{P}_X$ is stable by τ_X, we have, for all $p \in \mathcal{P}_X$ and all $g \in G$:

$$(p \circ f) \circ \tau_Y(g) = (p \circ \tau_X(g)) \circ f \in \mathcal{P}_Y.$$

Now, for every $\xi \in F_Y(\mathcal{P}_Y)$ and every $g \in G$, we have :

$$\|\xi \circ \Phi_{\tau_Y(g)}\|_{\mathcal{P}_Y} = \|T(\xi) \circ \Phi_{\tau_X(g)}\|_{F_X(\mathcal{P}_X)} = \|T(\xi)\|_{F_X(\mathcal{P}_X)} = \|\xi\|_{\mathcal{P}_Y}.$$

It follows that G acts by automorphisms on $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$.

Moreover, we have, for every $y \in Y$ and every $g \in G$, $d_Y(\tau_Y(g)y, y) = d_X(\tau_X(g)f(y), f(y))$, where d_X and d_Y are the labelled partitions pseudo-metric on respectively X and Y. Hence, for $y \in Y$, $y \to \tau_Y(g)y$ is continuous from G to (Y, d_Y). $\square$

Definition 1.3.15. *Let $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ be a space with labelled partitions, Y be a set and $f : Y \to X$ be a map. The structure of space with labelled partitions $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ given by Lemma 1.3.14 is called the* pull back by f of the space with labelled partitions $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$.

1.3.2 Examples

1. Spaces with measured walls

Our first example of spaces with labelled partitions is given by spaces with measured walls. Here we cite the definition of the structure of space with measured walls from [CSV12].

Let X be a set. We endow 2^X with the product topology and we consider, for $x \in X$, the clopen subset of 2^X, $\mathcal{A}_x := \{A \subset X \mid x \in A\}$.

Definition 1.3.16. *A* measured walls structure *is a pair (X, μ) where X is a set and μ is a Borel measure on 2^X such that for all $x, y \in X$:*

$$d_\mu(x, y) := \mu(\mathcal{A}_x \triangle \mathcal{A}_y) < +\infty$$

Proposition 1.3.17. *Let (X, μ) be a measured space with walls. Then, for every real number $q \geq 1$, $(X, \mathcal{P}, L^q(\mathcal{P}, \mu))$ is a space with labelled partitions where $\mathcal{P} = \{\mathbb{1}_h \mid h \in 2^X\}$.*
Morever, we have, for $x, y \in X$,

$$\|c(x, y)\|_q^q = d_\mu(x, y).$$

Proof. We denote $\mathcal{P} = \{\mathbb{1}_h \mid h \in 2^X\}$. Then $\mathcal{P}$ is a family of labelling functions on X and we denote by c the separation map of X associated with $\mathcal{P}$.
Let $x, y \in X$. For $h \in 2^X$, we have :

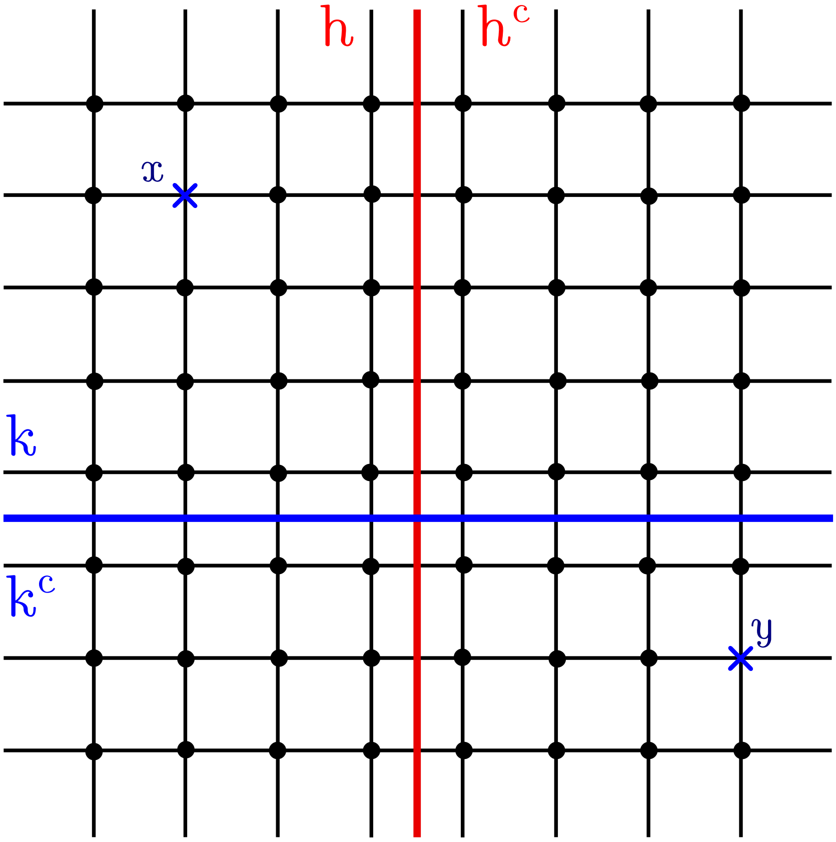
$$c(x, y)(\mathbb{1}_h) = \mathbb{1}_h(x) - \mathbb{1}_h(y) = \mathbb{1}_{\mathcal{A}_x}(h) - \mathbb{1}_{\mathcal{A}_y}(h).$$

The function $f : 2^X \to \mathcal{P}$ such that, for $h \in 2^X$, $f(h) = \mathbb{1}_h$ is a bijection, and we endow $\mathcal{P}$

with the direct image topology induced by f. Then, $\mu^* : \mathcal{P} \to \mathbb{R}$ such that, for any Borel subset A of $\mathcal{P}$, $\mu^*(A) = \mu(f^{-1}(A))$ is a Borel measure on $\mathcal{P}$.
We have $\|c(x,y)\|_q^q = \int_{\mathcal{P}} |c(x,y)(p)|^q d\mu^*(p) = \int_{2^X} |\mathbb{1}_{\mathcal{A}_x}(h) - \mathbb{1}_{\mathcal{A}_y}(h)|^q d\mu(h) = \mu(\mathcal{A}_x \triangle \mathcal{A}_y)$, and then :

$$\|c(x,y)\|_q^q = d_\mu(x,y) < +\infty.$$

It follows that, for all $x, y \in X$, $c(x,y)$ belongs to $L^q(\mathcal{P},\mu)$ and hence, $(X, \mathcal{P}, L^q(\mathcal{P},\mu))$ is a space with labelled partitions. $\square$



Examples of walls in $\mathbb{Z}^2$.

2. Gromov hyperbolic groups

The following Lemma is a reformulation of a result of Yu (see [Yu05], Corollary 3.2) based on a construction of Mineyev in [Min01].
For a triple x, y, z in a metric space (X, d), we denote by $(x|y)_z = \frac{1}{2}(d(x,z) + d(y,z) - d(x,y))$.

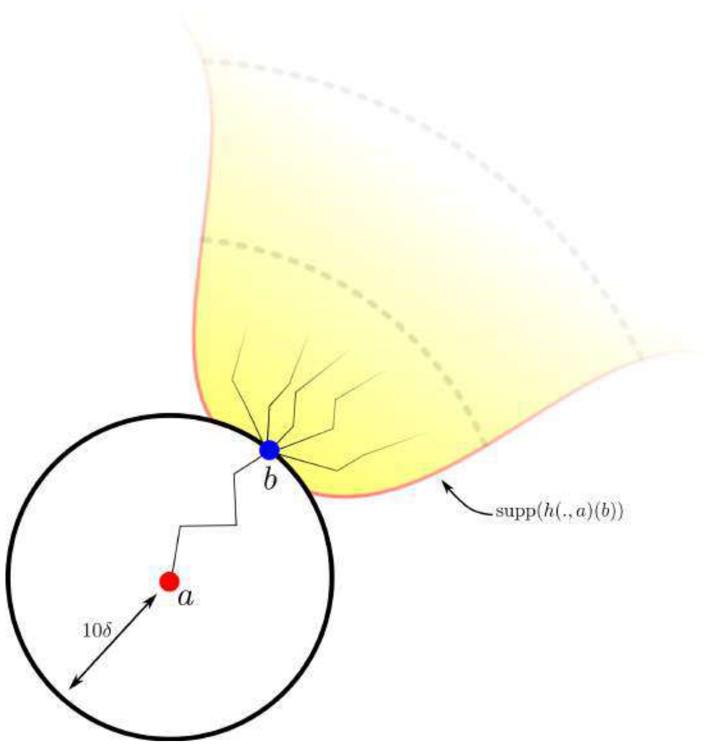
Lemma 1.3.18 (Mineyev, Yu)**.** *Let Γ be a finitely generated δ-hyperbolic group. Then there exists a Γ-equivariant function $h : \Gamma \times \Gamma \to \mathcal{F}_c(\Gamma)$ where $\mathcal{F}_c(\Gamma) = \{f : \Gamma \to \mathbb{R} \text{ with finite support} \mid \|f\|_1 = 1\}$ such that :*

1. *for all $a, x \in \Gamma$, $\mathrm{supp}\, h(x, a) \subset B(a, 10\delta)$,*
2. *there exists constants $C \geq 0$ and $\varepsilon > 0$ such that, for all $x, x', a \in \Gamma$,*

$$\|h(x, a) - h(x', a)\|_1 \leq Ce^{-\varepsilon(x|x')_a},$$

3. *there exists a constant $K \geq 0$ such that, for all $x, x' \in \Gamma$ with $d(x, x')$ large enough,*

$$\#\{a \in \Gamma \mid \mathrm{supp}\, h(x, a) \cap \mathrm{supp}\, h(x', a) = \emptyset\} \geq d(x, x') - K.$$



Support of the labelling function associated with (a, b) with $d(a, b) = 10\delta$.

This Lemma gives us a way to build a structure of labelled partitions on Gromov hyperbolic groups :

Proposition 1.3.19 (Labelled partitions on a δ-hyperbolic group)**.** *Let Γ be a finitely generated δ-hyperbolic group and we denote $\mathcal{P} = \{(a,b) \in \Gamma \times \Gamma \mid d(a,b) \leq 10\delta\}$. There exists $q_0 \geq 1$ such that, for all $q > q_0$, $(\Gamma, \mathcal{P}, \ell^q(\mathcal{P}))$ is a space with labelled partitions.*

Remark 1.3.20. *Notice that, stated this way, $\mathcal{P}$ is not a set of labelling functions on Γ. Implicitely, we do the following identification :*

$$\{(a,b) \in \Gamma \times \Gamma \mid d(a,b) \leq 10\delta\} \sim \{x \mapsto h(a,x)(b) \mid (a,b) \in \Gamma^2 \textit{ with } d(a,b) \leq 10\delta\}.$$

In fact, $x \mapsto h(a,x)(b)$ is uniquely determined by the pair (a,b).

Proof of Proposition 1.3.19. We fix a finite generating set of Γ and we denote d the word metric associated with it (and such that Γ is Gromov hyperbolic of constant δ with respect to d). As Γ is uniformly locally finite, there exists a constant $k > 0$ such that, for all $r > 0$ and $x \in \Gamma$, $\#B(x,r) \leq k^r$.

Let ε be as in 2. Lemma 1.3.18 and set $q_0 = \frac{\ln(k)}{\varepsilon}$. Let $q > q_0$. Then for all $q > q_0$,

$$\sum_{n \in \mathbb{N}} k^n e^{-nq\varepsilon} < +\infty.$$

Let h be the function given by Lemma 1.3.18 and notice that, for $x, x', a \in \Gamma$, since $\#\mathrm{supp}(h(x,a)) \leq k$,

$$\|h(x,a) - h(x',a)\|_q \leq 2k^{\frac{1}{q}} \|h(x,a) - h(x',a)\|_1. \qquad (*)$$

As said in the previous remark, we can see $\mathcal{P}$ as a set of labelling functions on Γ using the function h : we set, for $(a,b) \in \mathcal{P}$ and $x \in \Gamma$,

$$(a,b)(x) := h(x,a)(b).$$

We denote by c the separation map associated with $\mathcal{P}$. We have, for $x, x' \in \Gamma$,

$$
\begin{aligned}
\|c(x,x')\|^q_{\ell^q(\mathcal{P})} &= \sum_{(a,b)\in\mathcal{P}} |h(x,a)(b)-h(x,a)(b)|^q,\\
&= \sum_{a\in\Gamma} \|h(x,a)-h(x,a)\|^q_q \text{ by 1. Lemma 1.3.18},\\
&\le \sum_{a\in\Gamma} 2^q k\|h(x,a)-h(x,a)\|^q_1 \text{ by } (*),\\
&\le (2C)^q k \sum_{a\in\Gamma} e^{-q\varepsilon(x|x')_a} \text{ by 2. Lemma 1.3.18},\\
&\le (2C)^q k \sum_{a\in\Gamma} e^{-q\varepsilon(d(x,a)-d(x,x'))},\\
&\le (2C)^q k \sum_{n\in\mathbb{N}} k^n e^{-q\varepsilon(n-d(x,x'))}, \text{ and hence, since } q>q_0:\\
\|c(x,x')\|^p_{\ell^p(\mathcal{P})} &\le (2C)^q e^{q\varepsilon d(x,x')} < +\infty
\end{aligned}
$$

Thus $c(x,x')$ belongs to $\ell^p(\mathcal{P})$ for all $x,x'\in\Gamma$. It follows that $(\Gamma,\mathcal{P},\ell^p(\mathcal{P}))$ is a space with labelled partitions. □

Proposition 1.3.21. *Let Γ be a finitely generated δ-hyperbolic group. Let $q_0\ge 1$ as in Proposition 1.3.19 and for $q>q_0$, let $(\Gamma,\mathcal{P},\ell^q(\mathcal{P}))$ be the space with labelled partitions given by Proposition 1.3.19. Then the action of Γ by left-translation on itself induces a proper action of Γ by automorphisms on $(\Gamma,\mathcal{P},\ell^q(\mathcal{P}))$.*

Proof. We keep the notations used in the proof of Proposition 1.3.19. We first show that Γ acts by automorphisms on $(\Gamma,\mathcal{P},\ell^q(\mathcal{P}))$. Let $\gamma,x\in\Gamma$ and $(a,b)\in\mathcal{P}$. Since h is Γ-equivariant, we have :

$$\Phi_\gamma((a,b))(x)=(a,b)(\gamma x)=h(\gamma x,a)(b)=h(x,\gamma^{-1}a)(\gamma^{-1}b)=(\gamma^{-1}a,\gamma^{-1}b)(x),$$

And hence,

$$\Phi_\gamma((a,b))=(\gamma^{-1}a,\gamma^{-1}b)\in\mathcal{P}.$$

Moreover, for $\xi\in\ell^q(\mathcal{P})$, we have :

$$
\begin{aligned}
\|\xi\circ\Phi_\gamma\|^q_{\ell^q(\mathcal{P})} &= \sum_{(a,b)\in\mathcal{P}} |\xi(\gamma^{-1}a,\gamma^{-1}b)|^q,\\
&= \sum_{(\gamma a,\gamma b)\in\mathcal{P}} |\xi(a,b)|^q,\\
&= \sum_{(a,b)\in\mathcal{P}} |\xi(a,b)|^q,\\
\|\xi\circ\Phi_\gamma\|^q_{\ell^q(\mathcal{P})} &= \|\xi\|^q_{\ell^q(\mathcal{P})}.
\end{aligned}
$$

It follows that Γ acts by automorphisms on $(\Gamma, \mathcal{P}, \ell^q(\mathcal{P}))$.

Now, consider the identity element e of Γ and let $\gamma \in \Gamma$.
We denote $A = \{a \in \Gamma \mid supp\, h(\gamma, a) \cap supp\, h(e, a) = \emptyset\}$. Notice that for every $x, a \in \Gamma$, $\|h(x, a)\|_q \geq \frac{1}{k}$. We have, by 3. Lemma 1.3.18, when $d(\gamma, e)$ is large enough :

$$\begin{aligned} \|c(\gamma, e)\|^q_{\ell^q(\mathcal{P})} &= \sum_{a \in \Gamma} \|h(\gamma, a) - h(e, a)\|^q_q, \\ &\geq \sum_{a \in A} \|h(\gamma, a) - h(e, a)\|^q_q \geq \sum_{a \in A} (\frac{2}{k})^q, \text{ since } \|h(x, a)\|_q \geq \tfrac{1}{k} \\ \|c(\gamma, e)\|^q_{\ell^q(\mathcal{P})} &\geq (\frac{2}{k})^q (d(\gamma, e) - K). \end{aligned}$$

And hence, when $\gamma \to \infty$ in Γ, we have : $\|c(\gamma, e)\|^q_{\ell^q(\mathcal{P})} \geq (\frac{2}{k})^q (d(\gamma, e) - K) \to +\infty$. $\square$

3. Labelled partitions on metric spaces

It turns out that any pseudo-metric spaces (X, d) can be realized as a space with labelled partitions $(X, \mathcal{P}, F(\mathcal{P}))$ with $F(\mathcal{P}) \simeq \ell^\infty(X)$ and such that the pseudo-metric of labelled partitions is exactly d :

Proposition 1.3.22. *Let (X, d) be a pseudo-metric space and consider the family of labelling functions on X :*

$$\mathcal{P} = \{p_z : x \mapsto d(x, z) \mid z \in X\}.$$

Then $(X, \mathcal{P}, \ell^\infty(\mathcal{P}))$ is a space with labelled partitions.
Moreover, for all $x, y \in X$,

$$d_{\mathcal{P}}(x, y) = d(x, y),$$

where $d_{\mathcal{P}}$ is the pseudo-metric of labelled partitions on X.

Proof. Let c be the separation map associated with $\mathcal{P} = \{p_z : x \mapsto d(x, z) \mid z \in X\}$. For $x, y \in X$ and $p_z \in \mathcal{P}$, we have :

$$c(x, y)(p_z) = p_z(x) - p_z(y) = d(x, z) - d(y, z) \leq d(x, y),$$

and, in particular, $c(x, y)(p_y) = d(x, y)$, then,

$$\|c(x, y)\|_\infty = \sup_{p_z \in \mathcal{P}} |c(x, y)(p_z)| = d(x, y).$$

Hence, $(X, \mathcal{P}, \ell^\infty(\mathcal{P}))$ is a space with labelled partitions and $d_{\mathcal{P}}(x,y) = \|c(x,y)\|_\infty = d(x,y)$. $\square$

This result motivates the study of structures of spaces labelled partitions on a pseudo-metric space X : can we find other Banach spaces than $\ell^\infty(X)$ which gives a realization of the pseudo-metric on X as a pseudo-metric of labelled partitions ?

A first element of answer is given by the case of the discrete metric on a set. On every set, we can define a structure of labelled partitions which gives the discrete metric on this set :

Proposition 1.3.23. *Let X be a set and $\mathcal{P} = \{\Delta_x \mid x \in X\}$ be the family of labelling functions where, for $x \in X$, $\Delta_x = 2^{-\frac{1}{q}}\delta_x$.*
Then, for every $q \geq 1$, $(X, \mathcal{P}, \ell^q(\mathcal{P}))$ is a space with labelled partitions.

Proof. We have, for $x, y, z \in X$ with $x \neq y$:

$$c(x,y)(\Delta_z) = \Delta_z(x) - \Delta_z(y) = \begin{cases} 0 \text{ if } z \notin \{x,y\} \\ \pm 2^{-\frac{1}{q}} \text{ otherwise.} \end{cases}$$

and then,

$$\|c(x,y)\|_q^q = \sum_{z \in X} |c(x,y)(\Delta_z)|^q = |c(x,y)(\Delta_x)|^q + |c(x,y)(\Delta_y)|^q = 1.$$

$\square$

Notice that the labelled partitions pseudo-metric d on X in this case is precisely the discrete metric on X i.e. $d(x,y) = 1$ for all $x, y \in X$, $x \neq y$.

Definition 1.3.24 (Naive ℓ^q space with labelled partitions)**.** *Let X be a set and $\mathcal{P} = \{\Delta_x \mid x \in X\}$.*
For $q \geq 1$, $(X, \mathcal{P}, \ell^q(\mathcal{P}))$ is called the naive ℓ^q space with labelled partitions of X.

Remark 1.3.25. *Let X be a set, $q \geq 1$ and G a group acting on X. Then G acts by automorphisms on the naive ℓ^q space with labelled partitions of X.*
In fact, if, for $g \in G$, we denote $\tau(g) : x \mapsto gx$, we have, for $z \in X$,

$$\Delta_z \circ \tau(g) = \Delta_{g^{-1}z} \in \mathcal{P},$$

and, for all $\xi \in \ell^q(\mathcal{P})$,

$$\|\xi \circ \Phi_{\tau(g)}\|_q^q = \sum_{x\in X} |\xi(\Delta_{gx})|^q = \sum_{g^{-1}x\in X} |\xi(\Delta_x)|^q = \sum_{x\in X} |\xi(\Delta_x)|^q = \|\xi\|_q^q.$$

4. Labelled partitions on Banach spaces

Every Banach space has a natural structure of space with labelled partitions and the metric of labelled partitions of this structure is exactly the metric induced by the norm. Let f be a $\mathbb{K}$-valued function on a set B and $k \in \mathbb{K}$. We denote $f + k := \{x \mapsto f(x) + k\}$.

Definition 1.3.26. *Let* B *be a Banach space and* B' *be its topological dual. The set :*

$$\mathcal{P} = \{f + k \mid f \in B',\ k \in \mathbb{K}\}$$

is called the natural family of labelling functions *on* B*.*
Let c *be the separation map on* B *associated with* $\mathcal{P}$*. We denote :*

$$\delta(\mathcal{P}) = \{c(x, x') \mid x, x' \in B\}.$$

Remark 1.3.27. *This definition and the fact that the natural family of labelling functions contains the constant functions are motivated by the following : as we shall see in Lemma 1.3.30, a* G*-action on a Banach space* B *by affine isometries induces an action of* G *on the natural family of labelling functions on* B*.*

Proposition 1.3.28. *Let* $(B, \|\cdot\|)$ *be a Banach space and* $\mathcal{P}$ *be its natural family of labelling functions. Then* $\delta(\mathcal{P})$ *is isomorph to* B *and* $(B, \mathcal{P}, \delta(\mathcal{P}))$ *is a space with labelled partitions where* $\delta(B)$ *is viewed as an isometric copy of* B*. Moreover, we have, for* $x, x' \in B$ *:*

$$d(x, x') = \|x - x'\|,$$

where d *is the pseudo-metric of labelled partitions on* $(B, \mathcal{P}, \delta(\mathcal{P}))$*.*

Proof. Let $\mathcal{P} := \{f + k \mid f \in B',\ k \in \mathbb{K}\}$ and let c be the separation map on B associated with $\mathcal{P}$. Notice that for all $x, x' \in B$, $c(x - x', 0) = c(x, 0) - c(x', 0) = c(x, x')$. Then the map $T : B \to \delta(B)$ such that $x \mapsto c(x, 0)$ is clearly a surjective linear operator. Now, we have $c(x, 0) = 0 \Leftrightarrow \forall f \in B',\ f(x) = 0$, and hence, by Hahn-Banach Theorem, T is injective. It follows that T is an isomorphism.
The quantity $\|c(x, x')\|_{\delta(\mathcal{P})} := \|x - x'\|$ defines a norm on $\delta(\mathcal{P})$ and hence, $(\delta(\mathcal{P}), \|.\|_{\delta(\mathcal{P})})$ is

a Banach space as T is an isometric isomorphism. It follows immediately that $(B, \mathcal{P}, \delta(\mathcal{P}))$ is a space with labelled partitions. $\square$

Definition 1.3.29. *Let B be a Banach space. The space with labelled partitions $(B, \mathcal{P}, \delta(\mathcal{P}))$ where $\mathcal{P} = \{f + k \mid f \in B',\ k \in \mathbb{K}\}$ and $\delta(\mathcal{P}) \simeq B$ is called the* natural structure of labelled partitions on *B.*

Lemma 1.3.30. *Let G be a topological group. Then a continuous isometric affine action of G on a Banach space B induces a continuous action of G by automorphisms on the natural space with labelled partitions $(B, \mathcal{P}, \delta(\mathcal{P}))$ on B.*

Proof. Let α be a continuous isometric affine action of G on a Banach space B with linear part π and translation part b. Let $(B, \mathcal{P}, \delta(\mathcal{P}))$ be the natural space with labelled partitions on B.
Notice that for all $f \in B'$, $f \circ \pi(g) \in B'$ since π is an isometric representation. Hence, for all $g \in G$ and $p = f + k \in \mathcal{P}$:

$$p \circ \alpha(g) = f \circ \alpha(g) + k = f \circ \pi(g) + (k + f(b(g))) \in \mathcal{P}.$$

We denote, for $g \in G$ and $p \in \mathcal{P}$, $\Phi_g(p) = p \circ \alpha(g)$. We have, for $g \in G$ and $c(x, x') \in \delta(\mathcal{P})$,

$$\begin{aligned}
\|c(x,x') \circ \Phi_g\|_{\delta(\mathcal{P})} &= \|c(\alpha(g)x, \alpha(g)x')\|_{\delta(\mathcal{P})}, \\
&= \|\alpha(g)x - \alpha(g)x'\|, \\
&= \|\pi(g)(x - x')\|, \\
&= \|x - x'\|, \\
\|c(x,x') \circ \Phi_g\|_{\delta(\mathcal{P})} &= \|c(x,x')\|_{\delta(\mathcal{P})}.
\end{aligned}$$

It follows that G acts by automorphisms on $(B, \mathcal{P}, \delta(\mathcal{P}))$ and this action is clearly continuous since $d(x, x') = \|x - x'\|$ where d is the pseudo-metric of labelled partitions. $\square$

1.3.3 Link with isometric affine actions on Banach spaces

In this section, we aim to prove the two statements of Theorem 1 which gives an analog of the equivalence between proper actions on spaces with measured walls and Haagerup property in terms of proper actions on spaces with labelled partitions and isometric affine actions on Banach spaces ; and more particularly in the case of L^p spaces, using Hardin's result about extension of isometries on closed subspaces of L^p spaces.

Theorem 1.
Let G be topological group.

1. *If G acts (resp. acts properly) continuously by affine isometries on a Banach space B then there exists a structure $(G, \mathcal{P}, F(\mathcal{P}))$ of space with labelled partitions on G such that G acts (resp. acts properly) continuously by automorphisms on $(G, \mathcal{P}, F(\mathcal{P}))$ via its left-action on itself. Moreover, there exists a linear isometric embedding $F(\mathcal{P}) \hookrightarrow B$.*
2. *If G acts (resp. acts properly) continuously by automorphisms on a space with labelled partitions $(X, \mathcal{P}, F(\mathcal{P}))$ then there exists a (resp. proper) continuous isometric affine action of G on a Banach space B. Moreover, B is a closed subspace of $F(\mathcal{P})$.*

Corollary 2.
Let $p \geq 1$ with $p \notin 2\mathbb{Z} \setminus \{2\}$ and G be a topological group. G has property PL^p if, and only if, G acts properly continuously by automorphisms on a space with labelled partitions $(X, \mathcal{P}, F(\mathcal{P}))$ where $F(\mathcal{P})$ is isometrically isomorph to a closed subspace of an L^p space.

Proof of Corollary 2. The direct implication follows immediately from 1) Theorem 1.
Now, assume G acts properly continuously by automorphisms on a space $(X, \mathcal{P}, F(\mathcal{P}))$ and $T : F(\mathcal{P}) \hookrightarrow L^p(X, \mu)$ is a linear isometric embedding.
By 2) Theorem 1, there is a proper continuous isometric affine action α of G on a closed subspace B of $F(\mathcal{P})$ with $\alpha(g) = \pi(g) + b(g)$. Thus, as T is a linear isometry, $T(B)$ is a closed subspace of $L^p(X, \mu)$ and α' such that $\alpha'(g) = T \circ \pi(g) \circ T^{-1} + T(b(g))$ is a continuous isometric affine action of G on $T(B)$. Then, by Corollary 1.2.10, G has property PL^p. □

1. Labelled partitions associated with an isometric affine action

In this part, we introduce the space with labelled partitions associated with a continuous isometric affine action of a topological group G and we give a proof of 1) Theorem 1 by defining an action of G by automorphisms on this structure.

Given a continuous isometric affine action on a Banach space, we consider the pullback of the natural structure of space with labelled partitions of the Banach space on the group itself :

Definition 1.3.31. *Let G be a topological group and α be a continuous isometric affine action of G on a Banach space $(B, \|.\|)$ with translation part $b : G \to B$. Consider the*

pullback $(G, \mathcal{P}_\alpha, F_\alpha(\mathcal{P}_\alpha))$ *by* b *of the natural space with labelled partitions* $(B, \mathcal{P}, \delta(\mathcal{P}))$ *on* B*, where* $\mathcal{P} = B'$ *and* $\delta(\mathcal{P}) \simeq B$*.*
The triple $(G, \mathcal{P}_\alpha, F_\alpha(\mathcal{P}_\alpha))$ *is called* the space with labelled partitions associated with α*.*
More precisely, we have :
$\mathcal{P}_\alpha = \{f \circ b + k \mid f \in B',\ k \in \mathbb{K}\}$ *;*
$F_\alpha(\mathcal{P}_\alpha) \simeq \overline{\mathrm{Vect}(b(G))}^{\|.\|}$ *;*

Remark 1.3.32. - *The linear map* $T : F_\alpha(\mathcal{P}_\alpha) \hookrightarrow B$ *such that* $T : c_\alpha(g, h) \mapsto b(g) - b(h)$ *is an isometric embedding, where* c_α *is the separation map on* G *associated with* $\mathcal{P}_\alpha$*.*
- *If the continuous isometric affine action* α *is linear i.e.* $b(G) = \{0\}$*, then the space* $(G, \mathcal{P}_\alpha, F_\alpha(\mathcal{P}_\alpha))$ *with labelled partitions associated with* α *is degenerated in the sense that the quotient metric space associated with* (G, d) *contains a single point,* $\mathcal{P}_\alpha$ *contains only the zero function from* G *to* $\mathbb{K}$ *and* $F_\alpha(\mathcal{P}_\alpha) = \{0\}$*.*

Proposition 1.3.33. *Let* G *be a topological group and* $(G, \mathcal{P}, F(\mathcal{P}))$ *be the space with labelled partitions associated with a continuous isometric affine action of* G *on a Banach space* B*.*
Then the action of G *on itself by left-translation induces a continuous action of* G *by automorphisms on* $(G, \mathcal{P}, F(\mathcal{P}))$*.*

Proof. Let α be a continuous isometric affine action of G on a Banach space B with translation part $b : G \to B$. By Lemma 1.3.30, G acts continuously on the natural space with labelled partitions on $(B, \mathcal{P}, \delta(\mathcal{P}))$ on B. Moreover, the map b is G-equivariant since we have, for $g, h \in G$, $b(gh) = \alpha(g)b(h)$. By Lemma 1.3.14, it follows that the G-action on itself by left-translation induces a continuous action by automorphisms on $(G, \mathcal{P}, F(\mathcal{P}))$. $\square$

Proof of 1) Theorem 1. Assume α is continuous isometric affine action of G on a Banach space $(B, \|.\|)$ with translation part b and let G.
By Proposition 1.3.33, the G-action by left-translation on itself induces a continuous action by automorphisms on the space with labelled partitions associated with α, $(G, \mathcal{P}_\alpha, F_\alpha(\mathcal{P}_\alpha))$.
Moreover, assume α is proper. Then, by Remark 1.3.32, we have :

$$d_\alpha(g, e) = \|b(g)\| \underset{g \to \infty}{\longrightarrow} +\infty,$$

and hence, the G-action by automorphisms on $(G, \mathcal{P}_\alpha, F_\alpha(\mathcal{P}_\alpha))$ is proper. $\square$

2. From actions on a space with labelled partitions to isometric affine actions

We prove here statement 2) of Theorem 1 by giving a (non-canonical) way to build a proper continuous isometric affine action on a Banach space given a proper continuous action by automorphisms on space with labelled partitions.

Lemma 1.3.34. *Let G be a topological group, $(X, \mathcal{P}, F(\mathcal{P}))$ be a space with labelled partitions and we denote $E = \mathrm{Vect}(c(x, y) \mid x, y \in X)$ where c is the separation map associated with $\mathcal{P}$.*
If G acts continuously by automorphisms on $(X, \mathcal{P}, F(\mathcal{P}))$, then, for all $x, y \in X$, $(g, h) \mapsto c(gx, hy)$ is continuous from $G \times G$ to E.

Proof. Consider on the subspace E of $F(\mathcal{P})$ the topology given by the norm $\|.\|$ of $F(\mathcal{P})$. If $X \times X$ is endowed with the product topology of (X, d), as said in Remark 1.3.6, $c : X \times X \to E$ is continuous and, since the G-action on X is strongly continuous, for all $x, y \in X$, $(g, h) \mapsto (gx, hy)$ is continuous. Then, by composition, for all $x, y \in X$, $(g, h) \mapsto c(gx, hy)$ is continuous. □

Proposition 1.3.35. *Let G be a topological group acting continuously by automorphisms on a space with labelled partitions $(X, \mathcal{P}, F(\mathcal{P}))$. Then there exists a continuous isometric affine action of G on a Banach subspace B of $F(\mathcal{P})$.*
More precisely, $B = \overline{\mathrm{Vect}(c(x, y) \mid x, y \in X)}^{\|.\|}$ where c is the separation map associated with $\mathcal{P}$ and $\|.\|$ is the norm of $F(\mathcal{P})$, and moreover, the linear part π and the translation part b of the affine action are given by, for a fixed $x_0 \in X$:

$$\pi(g)\xi = \xi \circ \Phi_{\tau(g)} \text{ for } g \in G \text{ and } \xi \in B;$$

and

$$b(g) = c(gx_0, x_0) \text{ for } g \in G.$$

Proof. Let τ be the G-action on X.
By Definition 1.3.7 and Remark 1.3.10, the map $\Phi_{\tau(g)} : \mathcal{P} \to \mathcal{P}$ such that $\Phi_{\tau(g)}(p) = p \circ \tau(g)$ induces a linear representation π of G on $F(\mathcal{P})$ given by, for $\xi \in F(\mathcal{P})$ and $g \in G$:

$$\pi(g)\xi = \xi \circ \Phi_{\tau(g)}.$$

By the second requirement of Definition 1.3.7, we have $\|\pi(g)\xi\| = \|\xi\|$. Thus, π is an isometric linear representation of G on $F(\mathcal{P})$.

Consider $E = \mathrm{Vect}(c(x,y) \mid x,y \in X)$. Then the Banach subspace $B = \overline{E}^{\|\cdot\|}$ of $F(\mathcal{P})$ is stable under π since $\pi(g)(c(x,y)) = c(gx,gy)$ for $x,y \in X$, $g \in G$. Let us show that the representation π of G on B is strongly continuous. Let $\xi = \sum_{i=1}^n \lambda_i c(x_i,y_i) \in E$. We have, for $g \in G$,

$$\pi(g)\xi = \xi \circ \Phi_{\tau(g)} = \sum_{i=1}^n \lambda_i c(gx_i, gy_i) \in E,$$

and, by Lemma 1.3.34, for every i, $g \mapsto c(gx_i, gy_i)$ is continuous.
Hence, $g \mapsto \sum_{i=1}^n \lambda_i c(gx_i,gy_i) = \pi(g)\xi$ is continuous. Finally, by density, for all $\xi \in B$, $g \mapsto \pi(g)\xi$ is continuous from G to B.

Now, let us define the translation part of the action. Fix $x_0 \in X$ and set, for all $g \in G$, $b(g) = c(gx_0, x_0) \in E$. We claim b is a continuous 1-cocycle relative to π ; indeed, we have, for $g \in G$, $x,y \in X$, $c(gx,gy) = c(x,y) \circ \Phi_{\tau(g)} = \pi(g)c(x,y)$ and then, for $g,h \in G$,

$$b(gh) = c(ghx_0, x_0) = c(ghx_0, gx_0) + c(gx_0, x_0) = \pi(g)b(h) + b(g).$$

The continuity of b follows immediatly from Lemma 1.3.34.
Hence, the morphism $\alpha : G \to \mathrm{Isom}(B) \cap \mathrm{Aff}(B)$ defined by, for all $g \in G$, $\xi \in B$, $\alpha(g)\xi = \pi(g)\xi + b(g)$ is a continuous isometric affine action of G on B. □

Remark 1.3.36. *In the case where G is discrete, we do not have to find a subspace of $F(\mathcal{P})$ on which the representation is strongly continuous ; then we have the following statement :*
If G discrete acts by automorphisms on $(X, \mathcal{P}, F(\mathcal{P}))$, then there exists an isometric affine action of G on $F(\mathcal{P})$.

Proof of 2) Theorem 1. Assume G acts *properly* continuously on a space with labelled partitions $(X, \mathcal{P}, F(\mathcal{P}))$.
Consider the action α on the Banach subspace $B = \overline{E}^{\|\cdot\|}$ given by prop 1.3.35, where $E = \mathrm{Vect}(c(x,y) \mid x,y \in X)$ and $\alpha(g)\xi = \pi(g)\xi + b(g)$, for $g \in G$, $\xi \in B$.
Then we have, if we denote by d the pseudo-metric of labelled partitions on X :

$$\|b(g)\| = \|c(gx_0, x_0)\|_{\mathcal{P}} = d(gx_0, x_0) \underset{g\to\infty}{\longrightarrow} \infty$$

since the action of G on $(X, \mathcal{P}, F(\mathcal{P}))$ is proper, and hence, α is a proper continuous isometric affine action of G on B. □

1.4 Labelled partitions on a direct sum

In this section, we define a space with labelled partitions on the direct sum of a countable family of spaces with labelled partitions and we build on it a proper action given by proper actions on each factor.

1.4.1 Natural space with labelled partitions on a direct sum

Given a family of space with labelled partitions, we give a natural construction of a space with labelled partitions on the direct sum of this family. A similar construction in the case of spaces with measured walls can be found in [CMV04].

Definition 1.4.1. *Let I be an index set, $(X_i)_{i\in I}$ be a family of non empty sets and fix $x_0 = (x_i^0)_{i\in I} \in \prod_{i\in I} X_i$.*
The direct sum of the family $(X_i)_{i\in I}$ relative to x_0 is defined by :

$$\overset{x_0}{\bigoplus_{i\in I}} X_i := \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i \mid x_i \neq x_i^0 \text{ for finitely many } i \in I \right\}.$$

For $i \in I$, we denote by $\pi^X_{X_i} : X \to X_i$ the canonical projection from the direct sum to the factor X_i.

For $x = (x_i)_{i\in I} \in {}^{x_0}\bigoplus_{i\in I} X_i$, the support *of x is the finite subset of I :*

$$supp(x) = \{i \in I \mid x_i \neq x_i^0\}.$$

Definition 1.4.2. *Let I be an index set, $((X_i, \mathcal{P}_i, F_i(\mathcal{P}_i)))_{i\in I}$ be a family of spaces with labelled partitions and fix $x_0 = (x_i^0)_{i\in I} \in \prod_{i\in I} X_i$. We denote $X = {}^{x_0}\bigoplus_{i\in I} X_i$.*
Let $i \in I$. For $p_i \in \mathcal{P}_i$, we define the labelling function $p_i^{\oplus_i} : X \to \mathbb{K}$ by :

$$p_i^{\oplus_i} = p_i \circ \pi^X_{X_i}.$$

i.e., for $x = (x_i)_{i\in I} \in X$, $p_i^{\oplus_i}(x) = p_i(x_i)$.

We denote $\mathcal{P}_i^{\oplus_i} = \{p_i^{\oplus_i} \mid p_i \in \mathcal{P}_i\}$, and we call the set

$$\mathcal{P}_X = \bigcup_{i\in I} \mathcal{P}_i^{\oplus_i}$$

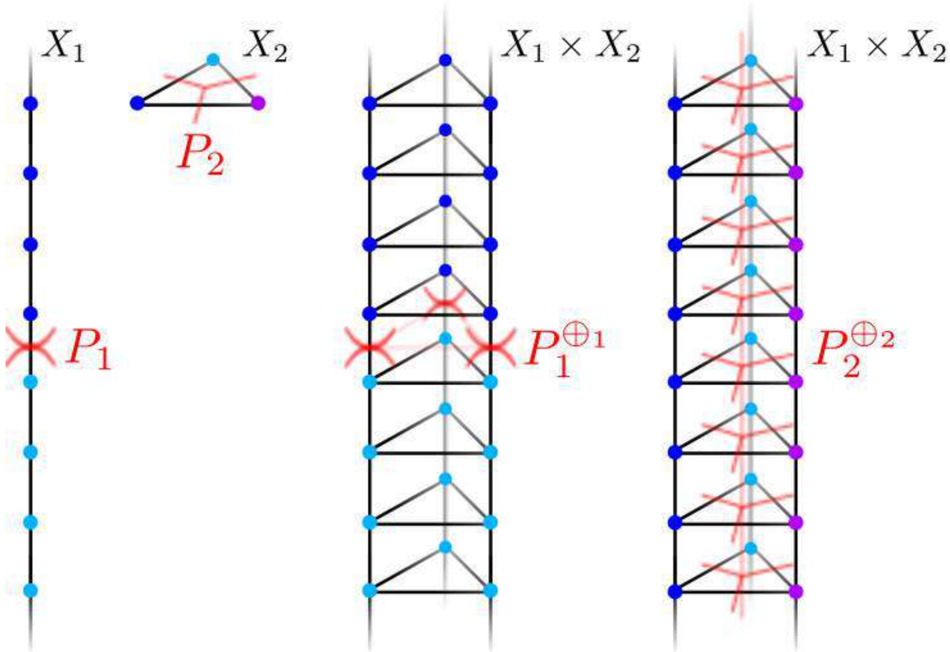
the natural family of labelling functions on X *(associated with the family* $(\mathcal{P}_i)_{i\in I}$*).*

Let X_1, X_2 be non empty sets and $\mathcal{P}_1, \mathcal{P}_2$ be families of labelling functions on, respectively, X_1 and X_2.
In terms of partitions, if P_1 is the partition of X_1 associated with $p_1 \in \mathcal{P}_1$, the partition $P_1^{\oplus_1}$ of $X_1 \times X_2$ associated with $p_1^{\oplus_1}$ is :

$$P_1^{\oplus_1} = \{h \times X_2 \mid h_1 \in P_1\},$$

and similarly, for $p_2 \in \mathcal{P}_2$, we have :

$$P_2^{\oplus_2} = \{X_1 \times k \mid k_1 \in P_2\}.$$



Partitions for the direct product

Definition 1.4.3. *Let I be a countable index set, $((X_i, \mathcal{P}_i, F_i(\mathcal{P}_i)))_{i\in I}$ be a family of spaces with labelled partitions and fix $x_0 = (x_i^0)_{i\in I} \in \prod_{i\in I} X_i$. We denote $X = {}^{x_0}\bigoplus_{i\in I} X_i$.*

Let $i \in I$. For $\xi_i \in F_i(\mathcal{P}_i)$, we denote $\xi_i^{\oplus_i} : \mathcal{P}_X \to \mathbb{K}$ the function :

$$\xi_i^{\oplus_i}(p) = \begin{cases} \xi_i(p_i) & \textit{if } p = p_i^{\oplus_i} \in \mathcal{P}_i^{\oplus_i} \\ 0 & \textit{if } p = p_j^{\oplus_j} \in \mathcal{P}_j^{\oplus_j} \textit{ with } i \neq j \end{cases}$$

Let $q \geq 1$. We denote $F_q(\mathcal{P}_X)$ the closure of

$$E_q(\mathcal{P}_X) := \left\{ \sum_{i \in I} \xi_i^{\oplus_i} \mid \xi_i \in F_i(\mathcal{P}_i) \text{ with } \xi_i \neq 0 \text{ for a finite number of } i \in I \right\},$$

endowed with the norm $\|.\|_{N_q}$ defined by, for $\xi = \sum_{i \in I} \xi_i^{\oplus_i}$:

$$\|\xi\|_{N_q} := \left(\sum_{i \in I} \|\xi_i\|^q_{F_i(\mathcal{P}_i)} \right)^{\frac{1}{q}}.$$

The vector space $F_q(\mathcal{P}_X)$ is called the q-space of functions on $\mathcal{P}_X$ of X.

Proposition 1.4.4. *Let I be a countable index set and $((X_i, \mathcal{P}_i, F_i(\mathcal{P}_i)))_{i \in I}$ be a family of spaces with labelled partitions and fix $x_0 = (x_i^0)_{i \in I} \in \prod_{i \in I} X_i$. We denote $X = {}^{x_0}\bigoplus_{i \in I} X_i$.*

Then $(F_q(\mathcal{P}_X), \|.\|_{N_q})$ is isometrically isomorph to $(\bigoplus^q_{i \in I} F_i(\mathcal{P}_i), \|.\|_q)$. In particular, $F_q(\mathcal{P}_X)$ is a Banach space.

1.4.2 Action on the natural space with labelled partitions of the direct sum

Let I be an index set and $(H_i)_{i \in I}$ be a family of groups. We denote $e_W = (e_{H_i})_{i \in I}$ where, for $i \in I$, e_{H_i} is the identity element of H_i.
We simply denote $\bigoplus\limits_{i \in I} H_i$ the group $W = {}^{e_W}\bigoplus\limits_{i \in I} H_i$ whose identity element is e_W.

Proposition 1.4.5.
Let I be a countable set and $(H_i)_{i \in I}$ be a family of groups such that, for each $i \in I$, H_i acts by automorphisms on a space with labelled partitions $(X_i, \mathcal{P}_i, F_i(\mathcal{P}_i))$. We denote $X = {}^{x_0}\bigoplus_{i \in I} X_i$ and $W = \bigoplus_{i \in I} H_i$.

Let $q \geq 1$. Then W acts by automorphisms on the natural space with labelled partitions on the direct sum $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$ via the natural action of W on X.

Proof. We denote by τ the W-action on X and for $w \in W$, $p \in \mathcal{P}_X$, $\Phi_{\tau(w)}(p) := p \circ \tau(w)$ and, for $i \in I$, we denote by τ_i the H_i-action on X and for $h_i \in H_i$, $p_i \in \mathcal{P}_i$, $\Phi_{\tau_i(h_i)}(p_i) := p_i \circ \tau_i(h_i)$.
Let $p \in \mathcal{P}_X = \bigcup_{i \in I} \mathcal{P}_i^{\oplus_i}$ and $w = (h_i)_{i \in I} \in W$. Then there exists $i \in I$ and $p_i \in \mathcal{P}_i$ such

that $p = p_i^{\oplus_i}$, and we have :

$$\Phi_{\tau(w)}(p_i^{\oplus_i}) = (\Phi_{\tau_i(h_i)}(p_i))^{\oplus_i} \in \mathcal{P}_i^{\oplus_i} \subset \mathcal{P}_X,$$

since $\Phi_{\tau_i(h_i)}(p_i)$ belongs to $\mathcal{P}_i$.

For $\xi = \sum_{i\in I} \xi_i^{\oplus_i} \in E_q(\mathcal{P}_X)$, we have :

$$\begin{aligned}
\xi \circ \Phi_{\tau(w)}(p) \quad &= \xi(p_i^{\oplus_i} \circ \tau(w)) \\
&= \xi((p_i \circ \tau_i(h_i))^{\oplus_i}) \\
&= \xi_i^{\oplus_i}((p_i \circ \tau_i(h_i))^{\oplus_i}) \\
&= \xi_i(p_i \circ \tau_i(h_i)) \\
&= \xi_i \circ \Phi_{\tau_i(h_i)}(p_i) \\
\xi \circ \Phi_{\tau(w)}(p) \quad &= (\xi_i \circ \Phi_{\tau_i(h_i)})^{\oplus_i}(p_i^{\oplus_i}),
\end{aligned}$$

And hence,

$$\xi \circ \Phi_{\tau(w)} = \sum_{i\in I} (\xi_i \circ \Phi_{\tau_i(h_i)})^{\oplus_i} \in F_q(\mathcal{P}_X).$$

By completeness of $F_q(\mathcal{P}_X)$, for all $\xi \in F_q(\mathcal{P}_X)$, $\xi \circ \Phi_{\tau(w)} \in F_q(\mathcal{P}_X)$.

Moreover, for $\xi = \sum_{i\in I} \xi_i^{\oplus_i} \in E_q(\mathcal{P}_X)$, we have :

$$\|\xi \circ \Phi_{\tau(w)}\|_{N_q}^q = \sum_{i\in I} \|\xi_i \circ \Phi_{\tau_i(h_i)}\|_{F_i(\mathcal{P}_i)}^q = \sum_{i\in I} \|\xi_i\|_{F_i(\mathcal{P}_i)} = \|\xi\|_{N_q}^q,$$

since, for all $i \in I$, $\|\xi_i \circ \Phi_{\tau_i(h_i)}\|_{F_i(\mathcal{P}_i)} = \|\xi_i\|_{F_i(\mathcal{P}_i)}$.

Thus, by density of $E_q(\mathcal{P}_X)$ in $F_q(\mathcal{P}_X)$, for all $\xi \in F_q(\mathcal{P}_X)$, $\|\xi \circ \Phi_{\tau(w)}\|_{N_q} = \|\xi\|_{N_q}$.

It follows that W acts by automorphisms on $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$.

$\square$

When I is finite, $X = {}^{x_0}\bigoplus_{i\in I} X_i$ is simply the direct sum of the X_i and does not depend on x_0. In this case, proper continuous actions on each factor $(X_i, \mathcal{P}_i, F_i(\mathcal{P}_i))$ induce a proper continuous action on the natural space with labelled partitions of the direct sum $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$:

Proposition 1.4.6. *Let $n \in \mathbb{N}^*$. For $i \in I = \{1, ..., n\}$, let H_i be a topological group acting* properly continuously *on a space with labelled partitions $(X_i, \mathcal{P}_i, F_i(\mathcal{P}_i))$; we denote $X = X_1 \times ... \times X_n$ and $W = H_1 \times ... \times H_n$.*

Let $q \geq 1$. Then W acts properly continuously *by automorphisms on the natural space with labelled partitions of the direct product $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$ via the natural action of W on X.*

Proof. We denote by c the separation map associated with $\mathcal{P}_X$ and, for $i \in I$, c_i the separation map associated with $\mathcal{P}_i$. By Proposition 1.4.5, W acts by automorphisms on $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$ and since I is finite, notice that $w = (h_i) \to \infty$ in W if, and only if, there exists $j \in I$ such that $h_j \to \infty$ in H_j.
Thus, for any $x = (x_1, ..., x_n) \in X$, we have, for $w = (h_i) \in W$ and any $j \in I$:

$$\|c(wx, x)\|_{N_q}^q = \sum_{i=1}^{n} \|c(h_i x_i, x_i)\|_{F_i(\mathcal{P}_i)}^q \geq \|c(h_j x_j, x_j)\|_{F_j(\mathcal{P}_j)}^q.$$

It follows that, when $w \to \infty$ in W, $\|c(wx, x)\|_{N_q} \to +\infty$ and then W acts properly on $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$. It remains to prove that the W-action on (X, d) is strongly continuous where W is endowed with the product topology of the H_i's. Remark that $d = (\sum_{i=0}^{n} d_i^q)^{\frac{1}{q}}$, then, the topology of (X, d) is equivalent to the product topology of the X_i's on X.
Let $x = (x_i)_{i \in I} \in X$. We denote by $\tau_x : W \to X$ the function $w \mapsto wx$. For all $i \in I$, $\pi_{X_i}^X \circ \tau_x : w \to h_i x_i$ is continuous since $h_i \to h_i x_i$ is continuous ; hence it follows that τ_x is continuous. $\square$

If I is countably infinite, even if each H_i-action on $(X_i, \mathcal{P}_i, F_i(\mathcal{P}_i))$ is proper, W does not act properly on the natural space with labelled partitions on the direct sum $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$ in general. In fact, let C be a positive real constant, and assume there exists, in each H_i, an element h_i such that $\|c_i(h_i x_i^0, x_i^0)\|_{F_i(\mathcal{P}_i)} \leq C$. For $j \in I$, the element $\delta_j(h_j)$ of W such that $\pi_{H_i}^W(\delta_j(h_j)) = e_{H_i}$ if $i \neq j$ and $\pi_{H_j}^W(\delta_j(h_j)) = h_j$ leaves every finite set of W when j leaves every finite set of I, but :

$$\|c(\delta_j(h_j)x_0, x_0)\|_{F_q(\mathcal{P}_X)} = \|c_i((h_j)x_i^0, x_i^0)\|_{F_i(\mathcal{P}_i)} \leq C.$$

And then, W does not act properly on $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$.
To make W act properly on a space with labelled partitions in the case where W is endowed with the discrete topology, we have to define a structure of labelled partitions on W such that the labelled partitions metric between e_W and w goes to infinity when the support of w leaves every finite set in I. To build this structure, we scale every labelling function of the naive ℓ^q space with labelled partitions on each factor H_i by a weight depending on i which grows as i leaves every finite set in I.

Notation 1.4.7. *Let I be a countable index set and $X = {}^{x_0}\bigoplus_{i\in I} X_i$ be a direct sum of sets X_i's.*

We say that, for $x \in X$, $supp(x)$ leaves every finite set in *I or $supp(x) \to \infty$* in *I if there exists $j \in supp(x)$ which leaves every finite set in I.*

Proposition 1.4.8. *Let I be a countable index set and $X = {}^{x_0}\bigoplus_{i\in I} X_i$ be a direct sum of countable sets X_i's. Then, an element $x = (x_i)_{i\in I} \in X$ leaves every finite set in X if either there exists $j \in I$ such that x_j leaves every finite set in X_j or $supp(x)$ leaves every finite set in I.*

Definition 1.4.9. *Let X be a set and w be a non-negative real.*
We set, for $x \in X$:

$$^{(w)}\Delta_x := 2^{-\frac{1}{q}} w\delta_x : X \to \mathbb{K},$$

where $\delta_x : X \to \{0,1\}$ is the Dirac function at x, and we call the set

$$^{(w)}\Delta := \{^{(w)}\Delta_x \mid x \in X\},$$

the w-weighted naive family of labelling functions *on X.*

Proposition 1.4.10. *Let X be a set and w be a non-negative real.*
Let $q \geq 1$. Then the triple $(X, {}^{(w)}\Delta, \ell^q({}^{(w)}\Delta))$ is a space with labelled partitions.
Moreover, if a group H acts on X, then H acts by automorphisms on $(X, {}^{(w)}\Delta, \ell^q({}^{(w)}\Delta))$.

Proof. It is a straightfoward generalization of Proposition 1.3.23 and Remark 1.3.25. □

Subsquently, for a countably infinite set I, we consider a function $\phi : I \to \mathbb{R}_+$ such that $\phi(i) \underset{i\to\infty}{\longrightarrow} +\infty$ (such a function always exists when I is countably infinite : for instance, take any bijective enumeration function ϕ from I to $\mathbb{N}$).

Lemma 1.4.11. *Let I be a countably infinite set and $(H_i)_{i\in I}$ be a family of countable discrete groups and we denote W the group $\bigoplus_{i\in I} H_i$ endowed with the discrete topology. Consider, on each H_i, the $\phi(i)$-weighted naive family of labelling functions ${}^{(\phi(i))}\Delta$ and we denote by ${}^{(\phi)}\Delta = \bigcup_{i\in I} {}^{(\phi(i))}\Delta^{\oplus_i}$ the natural set of labelling functions associated with $({}^{(\phi(i))}\Delta)_{i\in I}$.*

Let $q \geq 1$. Then, W acts by automorphisms on the natural space with labelled partitions

on the direct sum $(W, {}^{(\phi)}\Delta, F_q({}^{(\phi)}\Delta))$.
Moreover, we have :

$$\|c_\phi(w, e_W)\|_{F_q({}^{(\phi)}\Delta)} \to +\infty \text{ when } supp(w) \to \infty \text{ in } I,$$

where c_ϕ *is the separation map associated with* ${}^{(\phi)}\Delta$.

Proof. By Proposition 1.4.5, W acts by automorphisms on $(W, {}^{(\phi)}\Delta, F_q({}^{(\phi)}\Delta))$ and we have, for $w = (h_i), w' = (h'_i) \in W$:

$$\begin{aligned}\|c_\phi(w, w')\|^q_{F_q({}^{(\phi)}\Delta)} &= \sum_{i \in I} \|c_{\phi(i)}(h_i, h'_i)\|^q_q \\ &= \sum_{i \in \mathrm{supp}(w^{-1}w')} \phi(i)^q.\end{aligned}$$

Let $w \in W$ such that $\mathrm{supp}(w) \to \infty$ in I. Then there exists $j \in \mathrm{supp}(w)$ such that $j \to \infty$ in I and hence :

$$\|c_\phi(w, e_W)\|^q_{F_q({}^{(\phi)}\Delta)} = \sum_{i \in \mathrm{supp}(w)} \phi(i)^q \geq \phi(j)^q \to +\infty.$$

□

Proposition 1.4.12. *Let I be a countably infinite set and $(H_i)_{i \in I}$ be a family of countable discrete groups such that, for each $i \in I$, H_i acts* properly *by automorphisms on a space with labelled partitions $(X_i, \mathcal{P}_i, F_i(\mathcal{P}_i))$. We denote $X = {}^{x_0}\bigoplus_{i \in I} X_i$ and $W = \bigoplus_{i \in I} H_i$ endowed with the discrete topology.*

Let $q \geq 1$. Then there exists a structure of space with labelled partitions $(Y, \mathcal{P}_Y, F_q(\mathcal{P}_Y))$ on which W acts properly *by automorphisms.*

More precisely, $(Y, \mathcal{P}_Y, F(\mathcal{P}_Y))$ is the natural space with labelled partitions on the direct product $Y = X \times W$ where :

- *on X, we consider the natural space with labelled partitions on the direct sum of the family $((X_i, \mathcal{P}_i, F_i(\mathcal{P}_i)))_{i \in I}$;*
- *on W, we consider the natural space with labelled partitions on the direct sum of the family $((H_i, {}^{(\phi(i))}\Delta, \ell^q({}^{(\phi(i))}\Delta)))_{i \in I}$ where for $i \in I$, ${}^{(\phi(i))}\Delta$ is the $\phi(i)$-weighted naive family of labelling functions on H_i.*

Proof. By Proposition 1.4.5, W acts by automorphisms on both $(X, \mathcal{P}_X, F_q(\mathcal{P}_X))$ and $(W, {}^{(\phi)}\Delta, \ell^q({}^{(\phi)}\Delta))$. We set $Y = X \times W$ and consider the natural space with labelled partitions $(Y, \mathcal{P}_Y, F_q(\mathcal{P}_Y))$ on the direct product where :

$$\mathcal{P} = \mathcal{P}_X^{\oplus_1} \cup {}^{(\phi)}\Delta^{\oplus_2},$$

and

$$F_q(\mathcal{P}) \simeq F_q(\mathcal{P}_X) \oplus \ell^q({}^{(\phi)}\Delta).$$

Then, by Proposition 1.4.5, $W \times W$ acts by automorphisms on $(Y, \mathcal{P}_Y, F_q(\mathcal{P}_Y))$ via the action $(w_1, w_2).(x, w) = (w_1.x, w_2 w)$. Hence, W acts by automorphisms on $(Y, \mathcal{P}_Y, F_q(\mathcal{P}_Y))$, where W is viewed as the diagonal subgroup $\{(w, w) \mid w \in W\} < W \times W$.

It remains to prove that the W-action on $(Y, \mathcal{P}_Y, F_q(\mathcal{P}_Y))$ is proper. We have, for $w = (h_i) \in W$:

$$\begin{aligned} \|c_{\mathcal{P}_Y}(w.(x_0, e_W), (x_0, e_W))\|^q_{F_q(\mathcal{P}_Y)} &= \|c_{\mathcal{P}_X}(w.x_0, x_0)\|^q_{F_q(\mathcal{P}_X)} + \|c_\phi(w, e_W)\|^q_q \\ &= \sum_{i \in \text{supp}(w)} \|c(h_i x_i^0, x_i^0)\|^q_{F_i(\mathcal{P}_i)} + \sum_{i \in \text{supp}(w)} \phi(i)^q. \end{aligned}$$

If there exists $j \in I$, such that $h_j \to \infty$ in H_j, then, since the H_j-action is proper, we have :

$$\|c_{\mathcal{P}_Y}(w.(x_0, e_W), (x_0, e_W))\|_{F_q(\mathcal{P}_Y)} \geq \sum_{i \in \text{supp}(w)} \|c(h_i x_i^0, x_i^0)\|_{F_i(\mathcal{P}_i)} \to +\infty,$$

and if $supp(w) \to \infty$ in I, by Lemma 1.4.11, we have :

$$\|c_{\mathcal{P}_Y}(w.(x_0, e_W), (x_0, e_W))\|_{F_q(\mathcal{P}_Y)} \geq \|c_\phi(w, e_W)\|_q \to +\infty.$$

Hence, by Proposition 1.4.8, we conclude that W acts properly on $(Y, \mathcal{P}_Y, F_q(\mathcal{P}_Y))$. $\square$

1.4.3 Action of a semi-direct product on a space with labelled partitions

Definition 1.4.13 (compatible action). *Let G_1, G_2 be groups and $\rho : G_2 \to Aut(G_1)$ be a morphism of groups.*
Consider a set X on which G_1 and G_2. We say that the G_2-action is compatible *with the G_1-action with respect to ρ if, for $g_1 \in G_1$, $g_2 \in G_2$, we have, for all $x \in X$:*

$$g_2 g_1 g_2^{-1} x = \rho(g_2)(g_1) x.$$

Example 1.4.14. *If $\rho : G_2 \to Aut(G_1)$ is a morphism, then the action ρ of G_2 on G_1 is compatible with the action of G_1 on itself by translation with respect to ρ.*

Theorem 3.
Let $(X_1, \mathcal{P}_1, F_1(\mathcal{P}_1))$,$(X_2, \mathcal{P}_2, F_2(\mathcal{P}_2))$ be spaces with labelled partitions and G_1, G_2 be topological groups acting continuously by automorphisms on, respectively, $(X_1, \mathcal{P}_1, F_1(\mathcal{P}_1))$ and $(X_2, \mathcal{P}_2, F_2(\mathcal{P}_2))$ via τ_1 and τ_2.
Let $\rho : G_2 \to Aut(G_1)$ be a morphism of groups such that $(g_1, g_2) \mapsto \rho(g_2)g_1$ is continuous for the product topology on $G_1 \times G_2$.

Assume that there exists a continuous action by automorphisms of $G_1 \rtimes_\rho G_2$ on X_1 which extends the G_1 action.

Then the semi-direct product $G_1 \rtimes_\rho G_2$ acts continuously by automorphisms on the natural structure of labelled partitions $(X_1 \times X_2, \mathcal{P}, F_q(\mathcal{P}))$ on the direct product of $X_1 \times X_2$.

Moreover, if, for $i = 1, 2$, G_i acts properly on $(X_i, \mathcal{P}_i, F_i(\mathcal{P}_i))$, then $G_1 \rtimes_\rho G_2$ acts properly on $(X_1 \times X_2, \mathcal{P}, F(\mathcal{P}))$.

Proof of Theorem 3. Let us denote by τ_1 the G_1-action on X_1, by τ_2 the G_2-action on X_2 and by $\tilde{\rho}$ the G_2-action on G_1 defined by the restriction on G_2 of the $G_1 \rtimes_\rho G_2$-action on X_1. Then $\tilde{\rho}$ is compatible with τ_1 with respect to ρ.
We denote by τ the action of $G = G_1 \rtimes_\rho G_2$ on $X = X_1 \times X_2$ defined by :

$$\tau(g_1, g_2)(x_1, x_2) = (\tau_1(g_1)(\tilde{\rho}(g_2)x_1), \tau_2(g_2)x_2).$$

We show that, via this action, G acts by automorphisms on the direct product of spaces with labelled partitions $(X, \mathcal{P}, F_q(\mathcal{P}))$ where $\mathcal{P} = \mathcal{P}_1^{\oplus_1} \cup \mathcal{P}_2^{\oplus_2}$ and $F_q(\mathcal{P}) \simeq F_1(\mathcal{P}_1) \oplus F_2(\mathcal{P}_2)$ endowed with the q-norm of the direct sum for $q \geq 1$.

Let $p \in \mathcal{P}$ and $g = (g_1, g_2) \in G$. If $p = p_1^{\oplus_1} \in \mathcal{P}_1^{\oplus_1}$, then, for all $x = (x_1, x_2) \in X$, we have :

$$\begin{aligned}
\Phi_{\tau(g)}(p)(x) \quad &= p(\tau(g)x) \\
&= p_1^{\oplus_1}(\tau_1(g_1)(\tilde{\rho}(g_2)x_1), \tau_2(g_2)x_2) \\
&= p_1(\tau_1(g_1)(\tilde{\rho}(g_2)x_1)) \\
&= p_1 \circ \tau_1(g_1) \circ \tilde{\rho}(g_2)(x_1) \\
\Phi_{\tau(g)}(p)(x) \quad &= (p_1 \circ \tau_1(g_1) \circ \tilde{\rho}(g_2))^{\oplus_1}(x_1, x_2),
\end{aligned}$$

and since G_1 acts by automorphisms on $(X_1, \mathcal{P}_1, F_1(\mathcal{P}_1))$ via τ_1, we have $p_1 \circ \tau_1(g_1) \in \mathcal{P}_1$, and G_2 acts by automorphisms on $(X_1, \mathcal{P}_1, F_1(\mathcal{P}_1))$ via $\tilde{\rho}$, then $p_1 \circ \tau_1(g_1) \circ \tilde{\rho}(g_2) \in \mathcal{P}_1$. Hence, $\Phi_{\tau(g)}(p) = (p_1 \circ \tau_1(g_1) \circ \tilde{\rho}(g_2))^{\oplus_1}$ belongs to $\mathcal{P}$.

For $p = p_2^{\oplus_2} \in \mathcal{P}_2^{\oplus_2}$, we have $\Phi_{\tau(g)}(p) = (p_2 \circ \tau_2(g_2))^{\oplus_2}$ which belongs to $\mathcal{P}$ since G_2 acts by automorphisms on $(X_2, \mathcal{P}_2, F_2(\mathcal{P}_2))$ via τ_2.
Then, for all $g \in G$ and all $p \in \mathcal{P}$,

$$\Phi_{\tau(g)}(p) = p \circ \tau(g) \in \mathcal{P}.$$

Let us fix some notations. We denote, for $g_1 \in G_1$, $g_2 \in G_2$:

- $\Phi^{(1)}_{\tau_1(g_1)} : \mathcal{P}_1 \to \mathcal{P}_1$ the map $\Phi^{(1)}_{\tau_1(g_1)}(p_1) = p_1 \circ \tau_1(g_1)$;
- $\Phi^{(\tilde{\rho})}_{\tilde{\rho}(g_2)} : \mathcal{P}_1 \to \mathcal{P}_1$ the map $\Phi^{(\tilde{\rho})}_{\tilde{\rho}(g_2)}(p_1) = p_1 \circ \tilde{\rho}(g_2)$;
- $\Phi^{(2)}_{\tau_2(g_2)} : \mathcal{P}_2 \to \mathcal{P}_2$ the map $\Phi^{(2)}_{\tau_2(g_2)}(p_2) = p_2 \circ \tau_2(g_2)$.

Let ξ be in $F(\mathcal{P})$ and $g = (g_1, g_2) \in G$. We have, for all $p_1 \in \mathcal{P}_1$ and all $p_2 \in \mathcal{P}_2$:

$$\xi \circ \Phi_{\tau(g)}(p_1^{\oplus_1}) = (\xi_1 \circ \Phi^{(\tilde{\rho})}_{\tilde{\rho}(g_2)} \circ \Phi^{(1)}_{\tau_1(g_1)})^{\oplus_1}(p_1^{\oplus_1}),$$

and

$$\xi \circ \Phi_{\tau(g)}(p_2^{\oplus_2}) = (\xi_1 \circ \Phi^{(2)}_{\tau_2(g_2)})^{\oplus_2}(p_2^{\oplus_2}).$$

Hence, $\xi \circ \Phi_{\tau(g)} = (\xi_1 \circ \Phi^{(\tilde{\rho})}_{\tilde{\rho}(g_2)} \circ \Phi^{(1)}_{\tau_1(g_1)})^{\oplus_1} + (\xi_2 \circ \Phi^{(2)}_{\tau_2(g_2)})^{\oplus_2}$ and we have :

$$\begin{aligned}
\|\xi \circ \Phi_{\tau(g)}\|^q_{N_q} &= \|\xi_1 \circ \Phi^{(\tilde{\rho})}_{\tilde{\rho}(g_2)} \circ \Phi^{(1)}_{\tau_1(g_1)}\|^q_{F_1(\mathcal{P}_1)} + \|\xi_2 \circ \Phi^{(2)}_{\tau_2(g_2)}\|^q_{F_2(\mathcal{P}_2)} \\
&= \|\xi_1\|^q_{F_1(\mathcal{P}_1)} + \|\xi_2\||^q_{F_2(\mathcal{P}_2)} \\
\|\xi \circ \Phi_{\tau(g)}\|_{N_q} &= \|\xi\|_{N_q}
\end{aligned}$$

It follows that $G_1 \rtimes_\rho G_2$ acts by automorphisms on the space with labelled partitions $(X_1 \times X_2, \mathcal{P}, F_q(\mathcal{P}))$.

It remains to check this action by automorphisms is continuous, i.e. for all $x \in X$, $g \mapsto \tau(g)x$ is continuous.

As a set $G_1 \rtimes_\rho G_2$ is simply $G_1 \times G_2$ and since $(g_1, g_2) \mapsto \rho(g_2)g_1$ is continuous, the product topology on $G_1 \times G_2$ is compatible with the group structure of $G_1 \rtimes_\rho G_2$ (see [Bou71], III.18 Proposition 20).

Moreover, τ_1, τ_2 and $\tilde{\rho}$ are strongly continuous, then, for all $(x_1, x_2) \in X$, the map $(g_1, g_2) \to (\tau(g_1)(\tilde{\rho}(g_2)x_1), \tau_2(g_2)x_2)$ is continuous from $G_1 \times G_2$ endowed with the product

topology to (X, d) where d is the labelled partitions pseudo-metric.
Hence, $G_1 \rtimes_\rho G_2$ acts continuously by automorphisms on $(X, \mathcal{P}, F_q(\mathcal{P}))$.

Assume, for $i = 1, 2$, G_i acts properly on $(X_i, \mathcal{P}_i, F_i(\mathcal{P}_i))$ via τ_i, and we denote by c_i the separation map associated with $\mathcal{P}_i$.
Fix $x_0 = (x_1, x_2) \in X_1 \times X_2$.
The following egality holds for every $g = (g_1, g_2) \in G_1 \rtimes_\rho G_2$:

$$\|c(\tau(g)x_0, x_0)\|^q_{N_q} = \|c_1(\tau_1(g_1)(\tilde{\rho}(g_2)x_1), x_1)\|^q_{F_1(\mathcal{P}_1)} + \|c_2(\tau_2(g_2)x_2, x_2)\|^q_{F_2(\mathcal{P}_2)}.$$

Since $G_1 \rtimes_\rho G_2$ is endowed with the product topology of G_1 and G_2, $g = (g_1, g_2) \to \infty$ in $G_1 \rtimes_\rho G_2$ if, and only if, $g_1 \to \infty$ in G_1 or $g_2 \to \infty$ in G_2. Hence, we have two disjoint cases :

First case : $g_1 \to \infty$ in G_1 and g_2 belongs to a compact subset K_2 of G_2.
By continuity of $g_2' \mapsto \|c(\tilde{\rho}(g_2')x_1, x_1)\|_{F_1(\mathcal{P}_1)}$, there exists $C(K_2) \geq 0$ such that, for every $g_2' \in K_2$, $\|c(\tilde{\rho}(g_2')x_1, x_1)\|_{F_1(\mathcal{P}_1)} \leq C(K_2)$, and, hence,

$$\begin{aligned} \|c(\tau(g_1)\tilde{\rho}(g_2)x_1, \tilde{\rho}(g_2)x_1)\|_{F_1(\mathcal{P}_1)} &\leq \|c(\tau_1(g_1)\tilde{\rho}(g_2)x_1, x_1)\|_{F_1(\mathcal{P}_1)} + \|c(\tilde{\rho}(g_2)x_1, x_1)\|_{F_1(\mathcal{P}_1)} \\ &\leq \|c(\tau(g_1)\tilde{\rho}(g_2)x_1, x_1)\|_{F_1(\mathcal{P}_1)} + C(K_2). \end{aligned}$$

But, since G_1 acts properly on $(X_1, \mathcal{P}_1, F_1(\mathcal{P}_1))$, $\|c(\tau(g_1)\tilde{\rho}(g_2)x_1, \tilde{\rho}(g_2)x_1)\|_{F_1(\mathcal{P}_1)} \underset{g_1 \to \infty}{\longrightarrow} +\infty$, and then,

$$\|c(\tau(g_1)\tilde{\rho}(g_2)x_1, x_1)\|_{F_1(\mathcal{P}_1)} \underset{g_1 \to \infty}{\longrightarrow} +\infty.$$

It follows that $\|c(\tau(g)x_0, x_0)\|_{N_q} \underset{g_1 \to \infty}{\longrightarrow} +\infty$.

Second case : $g_2 \to \infty$ in G_2.
We have $\|c_2(\tau_2(g_2)x_2, x_2)\|_{F_2(\mathcal{P}_2)} \underset{g_2 \to \infty}{\longrightarrow} +\infty$ and then $\|c(\tau(g)x_0, x_0)\|_{N_q} \to +\infty$.
Finally, as required, we have

$$\|c(\tau(g)x_0, x_0)\|_{N_q} \underset{g \to \infty}{\longrightarrow} +\infty,$$

and then, $G_1 \rtimes_\rho G_2$ acts properly by automorphisms on $(X, \mathcal{P}, F_q(\mathcal{P}))$.

□

1.5 Wreath products and property PL^p

Using Theorem 3, we simplify a part of the proof of Th 6.2 in [CSV12] where Cornulier, Stalder and Valette establish the stability of the Haagerup property by wreath product ; and we generalize it in the following way : the wreath product of a group with property PL^p by a Haagerup group has property PL^p.

Theorem 4.
Let H, G be countable discrete groups, L be a subgroup of G and $p > 1$, with $p \notin 2\mathbb{Z} \setminus \{2\}$. We denote by I the quotient G/L and $W = \bigoplus_I H$. Assume that G is Haagerup, L is co-Haagerup in G and H has property PL^p.
Then the permutational wreath product $H \wr_I G = W \rtimes G$ has property PL^p.

1.5.1 Permutational wreath product

We first introduce the notion of permutational wreath product :

Definition 1.5.1. *Let H, G be countable groups, I be a G-set and $W = \bigoplus_{i \in I} H$. The* permutational wreath product *$H \wr_I G$ is the group :*

$$H \wr_I G := W \rtimes_\rho G,$$

where G acts by shift on W via ρ i.e. $\rho(g) : (h_i)_{i \in I} \mapsto (h_{g^{-1}i})_{i \in I}$, for $g \in G$.
When $I = G$, $H \wr_G G$ is simply called wreath product *and is denoted $H \wr G$.*

1.5.2 Property PL^p for the permutational wreath product

To prove Theorem 4, we need the following structure of space with measured walls relative to the wreath product built in [CSV12], Theorem 4.2 (see [CSV12] § 6.1 for examples of co-Haagerup subgroups) :

Definition 1.5.2. *Let G be a group and L be subgroup of G. We say that L is* co-Haagerup *in G if there exists a proper G-invariant conditionally negative definite kernel on G/L.*

Theorem 1.5.3 (Cornulier, Stalder, Valette)**.** *Let H, G be countable discrete groups and let L be a subgroup of G. We denote by I the quotient G/L and $W = \bigoplus_I H$.*
Suppose that G is Haagerup and that L is co-Haagerup in G.
Then there exists a structure $(W \times I, \mu)$ of space with measured walls on $W \times I$, with wall

pseudo-metric denoted by d_μ*, on which* $W \rtimes G$ *acts by automorphisms and which satisfies, for any* $x_0 = (w_0, i_0) \in W \times I$ *and for all* $g \in G$ *:*

$$d_\mu((w,g)x_0, x_0) \to +\infty \text{ when } w \in W \text{ is such that } supp(w) \to \infty \text{ in } I.$$

Lemma 1.5.4. *Let* H, G *be countable discrete groups,* L *be a subgroup of* G *and* $q \geq 1$*. We denote by* I *the quotient* G/L *and* $W = \bigoplus_I H$*. Suppose that* G *is Haagerup,* L *is co-Haagerup in* G *and* H *has property* PL^q*.*
Then W *and* G *acts by automorphisms on a space* $(X, \mathcal{P}, F(\mathcal{P}))$ *with labelled partitions such that :*

- *the* W*-action is proper,*

- *the* G*-action is compatible with the* W*-action,*

- *the Banach space* $F(\mathcal{P})$ *is isometrically isomorph to a Banach subspace of a* L^q *space.*

Proof. Consider the $W \rtimes G$-action on the space with measured walls $(W \times I, \mu)$ given by Theorem 1.5.3. Then, by Proposition 1.3.17, $W \rtimes G$ acts by automorphisms on the space with labelled partitions $(W \times I, \mathcal{P}_\mu, L^q(\mathcal{P}_\mu, \mu))$. Let $y_0 = (e_W, i_0) \in W \times I$. The separation map c_μ associated with $\mathcal{P}_\mu$ satisfies :

$$\|c_\mu((w,g)y_0, y_0)\|_q = d_\mu((w,g)y_0, y_0).$$

Now, consider the structure of space with labelled partitions on H given by its proper isometric affine action on a space $L^q(E, \nu)$. By Proposition 1.4.5, W acts by automorphisms on the natural structure of space with labelled partitions $(W, \mathcal{P}_W, F_q(\mathcal{P}_W))$ of the direct sum of spaces with labelled partitions on H. Moreover, G acts by automorphisms on $(W, \mathcal{P}_W, F_q(\mathcal{P}_W))$ by shift via its action on I.
We denote $X = (W \times I) \times W$ and consider the space with labelled partitions $(X, \mathcal{P}, F(\mathcal{P}))$ given by the direct product of spaces with labelled partitions $(W \times I, \mathcal{P}_\mu, L^q(\mathcal{P}_\mu, \mu))$ and $(W, \mathcal{P}_W, F_q(\mathcal{P}_W))$. Then we have actions by automorphisms τ_W of W and τ_G on X given by, for $x = (w_1, i, w_2) \in X$, $w \in W$ and $g \in G$:

$$\tau_W(w)x = (ww_1, i, ww_2) \text{ and } \tau_G(g)x = (\rho(g)w_1, gi, \rho(g)w_2).$$

The action τ_G is clearly compatible with τ_W since $W \rtimes_\rho G$ acts naturally on W and on

$W \times I$.

The Banach space $F(\mathcal{P})$ is isometrically isomorph to the q-direct sum $L^q(\mathcal{P}_\mu, \mu) \oplus F_q(\mathcal{P}_W)$, then $F(\mathcal{P})$ is isometrically isomorph to a Banach subspace of $L^q(\mathcal{P}_\mu, \mu) \oplus (\bigoplus_I^q L^q(E, \nu))$. It follows that $F(\mathcal{P})$ is isometrically isomorph to a Banach subspace of a L^q space. We denote $x_0 = (e_W, i_0, e_W) \in X$. We have, for $w = (h_i)_{i \in I} \in W$:

$$\begin{aligned} \|c(\tau_W(w)x_0, x_0)\|^q_{F(\mathcal{P})} &= \|c_{\mathcal{P}_\mu}((w, i_0), (e_W, i_0))\|^q_{F_\mu(\mathcal{P}_\mu)} + \|c_{\mathcal{P}_W}(w, e_W)\|^q_{F_q(\mathcal{P}_W)} \\ &= d_\mu((w, e_G)y_0, y_0) + \sum_{i \in \mathrm{supp}(w)} \|c_{\mathcal{P}_H}(h_i, e_H)\|^q_{F_H(\mathcal{P}_H)} \end{aligned}$$

Hence, W acts properly by automorphisms on $(X, \mathcal{P}, F(\mathcal{P}))$: indeed, $w = (h_i) \to \infty$ in W if, and only if, $\mathrm{supp}(w) \to \infty$ in I or there exists $j \in I$ such that $h_j \to \infty$ in H ; then, in the first case, by the previous theorem, $d_\mu((w, e_G)y_0, y_0) \to +\infty$ and in the second case, $\sum_{i \in \mathrm{supp}(w)} \|c_{\mathcal{P}_H}(h_i, e_H)\|^q_{F_H(\mathcal{P}_H)} \geq \|c_{\mathcal{P}_H}(h_j, e_H)\|^q_{F_H(\mathcal{P}_H)} \to +\infty$.

□

Proof of Theorem 4. By Lemma 1.5.4, W and G act by automorphisms on a space $(X, \mathcal{P}, F(\mathcal{P}))$ with labelled partitions such that the W-action is proper, and the G-action is compatible with the W-action with respect to ρ. Moreover, since G is Haagerup, G acts properly by automorphisms on a space $(Y, \mathcal{P}', F'(\mathcal{P}'))$ with labelled partitions where $F'(\mathcal{P}')$ isometrically isomorph to a L^q space.

Hence, by Theorem 3, $H \wr_I G = W \rtimes_\rho G$ acts properly by automorphisms on a space $(Z, \mathcal{P}_Z, F_Z(\mathcal{P}_Z))$ where $F_Z(\mathcal{P}_Z)$ is isometrically isomorph to $F(\mathcal{P}) \oplus F'(\mathcal{P}')$ endowed with the q-norm of the direct sum. It follows that $F_Z(\mathcal{P}_Z)$ is isometrically isomorph to a Banach subspace of a L^q space.

Thus, by Corollary 2, $H \wr_I G$ has property PL^q. □

1.6 Free product of spaces with labelled partitions

In this section, we investigate labelled partitions on the free product of spaces with labelled partitions, given a notion of free product of non-empty sets introduced by Dreesen in [Dre11] which generalizes the free product of groups. We will be treating some other properties of free product of spaces and metric spaces in Section 2.6.2.

Given a set of symbols S, we denote by $\mathcal{M}(S)$ the set of words in S i.e. $\mathcal{M}(S) = \{s_1...s_n \mid s_i \in S,\ n \in \mathbb{N}\}$.

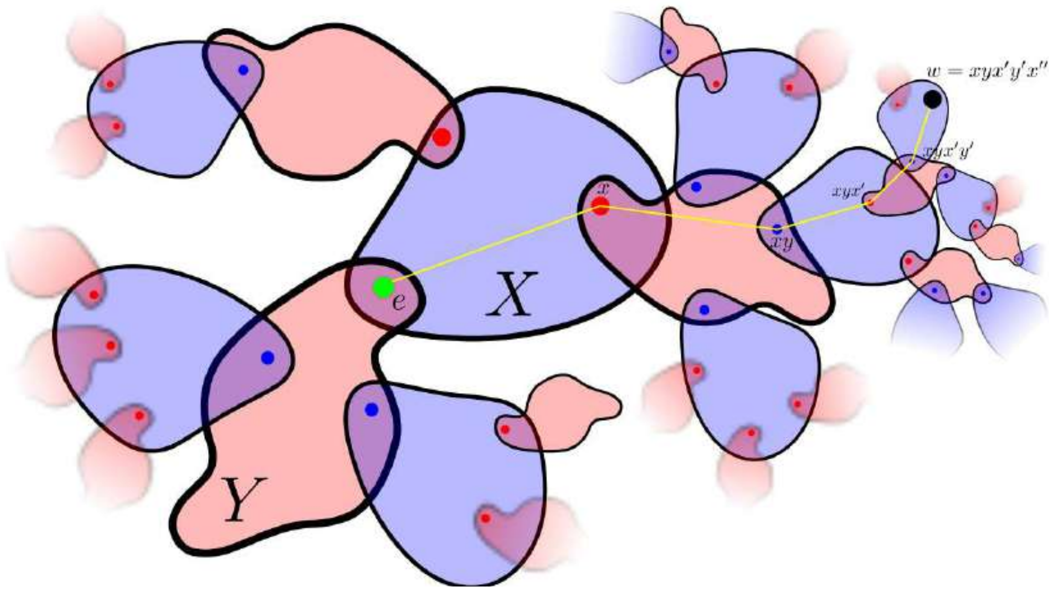
Definition 1.6.1. *Let S, S' be sets. An* alternating word *in S and S' is a word $w_1w_2...w_n \in \mathcal{M}(S \sqcup S')$ such that, for $i = 1, ..., n-1$, either $w_i \in S$ and $w_{i+1} \in S'$, or $w_i \in S'$ and $w_{i+1} \in S$.*
We denote $Alt(S, S')$ the set of all alternating words in S and S'.

Definition 1.6.2 (Free product of spaces)**.** *Let X, Y be non-empty sets and fix $x_0 \in X$, $y_0 \in Y$. The* free product *of X and Y on the basepoints x_0, y_0 is the set :*

$$X \underset{x_0 \sim y_0}{*} Y = Alt(X \smallsetminus \{x_0\}, Y \smallsetminus \{y_0\}).$$

When there is no ambiguity on the fixed points, we simply denote the free product of X and Y by $X * Y$.
We denote by e the empty word of $X * Y$.

The set $X \underset{x_0 \sim y_0}{*} Y$ can be visualized as two copies of X and Y, glued together by identifying x_0 and y_0 ; on each point but x_0 of the copy of X, glue a copy of Y by identifying this point with y_0 and similarly, on each point but y_0 of the first copy of Y, glue a copy of X by identifying this point with x_0 ; by induction, do the same process on every new copy of X and Y. Notice that only the first two copies are glued together at $x_0 \sim y_0$: this point corresponds to the empty word e in $X * Y$.



Definition 1.6.3 (Subword relation)**.** *Let $w, w' \in X * Y$. We say that w' is a (starting)*

subword of w and we denote $w' \leq w$ if there exists $u \in X * Y$ such that :

$$w = w'u.$$

The set of (starting) subwords in $X * Y$ of w is denoted by :

$$Sub(w) = \{w' \mid w' \leq w\},$$

and we set :

$$Sub^*(w) = Sub(w) \setminus \{w\}.$$

Remark 1.6.4. - *The relation "being a subword" $\leq$ is a partial order relation on $X * Y$ and the empty word e is a minimal element for $\leq$ in $X * Y$.*

- *For $w \in X * Y$, $Sub(w)$ is a totally ordered finite subset of $X * Y$ whose minimal element is e and maximal element is w.*

1.6.1 Left-coset projections

In this part, X, Y are non-empty sets, $x_0 \in X$, $y_0 \in Y$ are basepoints and $X * Y = X \underset{x_0 \sim y_0}{*} Y$.

We have the following partition of $X * Y$:

$$X * Y = \{e\} \sqcup E_X \sqcup E_Y,$$

where E_X is the set of alternating words whose last letter belongs to $X \setminus \{x_0\}$ and E_Y is the set of alternating words whose last letter belongs to $Y \setminus \{y_0\}$.

Definition 1.6.5 (Left-cosets)**.** *Let $w \in X * Y$.*

— *If $w \in E_Y$, the* left-X-coset associated with w *is the subset of $X * Y$:*

$$wX = \{w\} \cup \{wx \in X * Y \mid x \in X \setminus \{x_0\}\}.$$

— *If $w \in E_X$, the* left-Y-coset associated with w *is the subset of $X * Y$:*

$$wY = \{w\} \cup \{wy \in X * Y \mid y \in Y \setminus \{y_0\}\}.$$

Moreover, we also define the left-X-coset associated with e*, denoted eX and the* left-Y-coset associated with e, *denoted* eY *by, respectively :*

$$eX = \{\textit{words in } X * Y \textit{ of at most one letter in } X\},$$

and

$$eY = \{\textit{words in } X * Y \textit{ of at most one letter in } Y\}.$$

Remark 1.6.6. *There is an obvious bijection between the set of left-X-cosets and $E_Y \cup \{e\}$ since for $w, w' \in E_Y$, $wX = w'X$ if, and only if, $w = w'$. Similarly, the set of left-Y-cosets and $E_X \cup \{e\}$ are equipotent. Hence, if X and Y are countable the set of left-cosets is countable.*

Definition 1.6.7 (Left-coset projections)**.**

— *Let $w \in E_Y$. The map $\pi_{wX} : X * Y \to X$ defined, for $w' \in X * Y$, by :*

$$\pi_{wX}(w') = \begin{cases} x \textit{ if } w' = wxu \textit{ with } x \in X \textit{ and } u \in X * Y, \\ x_0 \textit{ if } w \textit{ is not a subword of } w'. \end{cases}$$

is called the wX-projection *of $X * Y$ on X.*

— *Let $w \in E_X$. The map $\pi_{wY} : X * Y \to Y$ defined, for $w' \in X * Y$, by :*

$$\pi_{wY}(w') = \begin{cases} y \textit{ if } w' = wyu \textit{ with } y \in Y \textit{ and } u \in X * Y, \\ y_0 \textit{ if } w \textit{ is not a subword of } w'. \end{cases}$$

is called the wY-projection *of $X * Y$ on Y.*

— *The map $\pi_{eX} : X * Y \to X$ defined by $\pi_{eX}(e) = x_0$ and for $w' \in X * Y$, $w' \neq e$:*

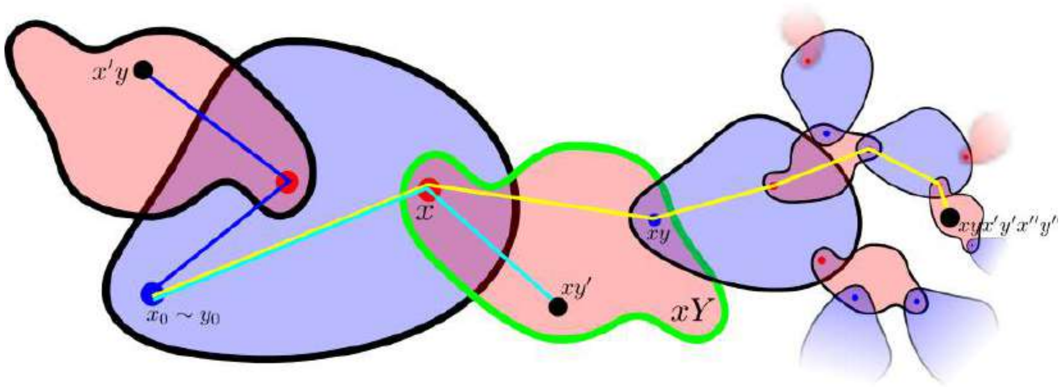
$$\pi_{eX}(w') = \begin{cases} x \textit{ if } w' = xu \textit{ with } x \in X \textit{ and } u \in X * Y, \\ x_0 \textit{ if the first letter of } w' \textit{ belongs to } Y \setminus \{y_0\}. \end{cases}$$

is called the eX-projection *of $X * Y$ on X.*

— *The map $\pi_{eY} : X * Y \to Y$ defined by $\pi_{eY}(e) = y_0$ and for $w' \in X * Y$, $w' \neq e$:*

$$\pi_{eY}(w') = \begin{cases} y \textit{ if } w' = yu \textit{ with } y \in Y \textit{ and } u \in X * Y, \\ y_0 \textit{ if the first letter of } w' \textit{ belongs to } X \setminus \{x_0\}. \end{cases}$$

is called the eY-projection *of $X * Y$ on Y.*



Example of left-coset projection : in this picture,
$\pi_{xY}(xyx'y'x''y'') = y$; $\pi_{xY}(xy') = y'$; $\pi_{xY}(x'y) = y_0$ and $\pi_{xY}(x) = y_0$.

Remark 1.6.8. *For every left-coset wZ, $\pi_{wZ}(e) = z_0$.*

Lemma 1.6.9. *Let $w' \in X * Y$. Then, $\pi_{wZ}(w') = z_0$ for all but finitely many left-cosets wZ of $X * Y$ where Z stands for X or Y and z_0 stands for x_0 or y_0 as appropriate.*

Proof. For $w \in X * Y$ with $w \neq e$, if $\pi_{wZ}(w') \neq z_0$ then w is a subword of w'. As w' admits finitely many subwords, it follows that $\pi_{wZ}(w') = z_0$ for all but finitely many left-cosets wZ. $\square$

Lemma 1.6.10. *Let $w', w'' \in X * Y$ with $w' \neq w''$. We have :*

$$\{w \mid \pi_{wZ}(w') \neq \pi_{wZ}(w'')\} = \{w_c\} \cup (Sub^*(w') \triangle Sub^*(w'')),$$

where w_c is the largest common subword of w' and w'' i.e. the maximal element of $Sub(w') \cap Sub(w'')$ for the order relation $\leq$.

Proof. Assume $w' \neq w''$. Consider the following decompositions of w' and w'' :

$$w' = w_c d' u_1 \ldots u_n \text{ and } w'' = w_c d'' v_1 \ldots v_m,$$

where w_c is the largest common subword of w' and w'' ; and d', d'' are the first differing letters (possibly x_0 or y_0) after w_c i.e. if $w_c \neq e$, $d' = \pi_{w_c Z}(w')$ and $d'' = \pi_{w_c Z}(w'')$ and if $w_c = e$, we set $d' = \pi_{eZ}(w')$, $d'' = \pi_{eZ}(w'')$ where Z is the common set of the first letters of w and w' if they belong to the same set, and $Z = X$ otherwise. As $w' \neq w''$, we have $d' \neq d''$. Notice that, as in the proof of Lemma 1.6.9, if $w \in X * Y$ is neither a subword of

w' nor w'', then $\pi_{wZ}(w') = z_0 = \pi_{wZ}(w'')$. It follows that :

$$\{w \in X * Y \mid \pi_{wZ}(w') \neq \pi_{wZ}(w'')\} \subset \mathrm{Sub}(w') \cup \mathrm{Sub}(w).$$

If $w \neq w_c$ is a common subword of w' and w'', then $w \in \mathrm{Sub}^*(w_c)$ i.e. there exists $z \in Z \smallsetminus \{z_0\}$ and $t \in X * Y$ such that $w_c = wzt$. Thus, $w' = wztd'u$ and $w'' = wztd''v$, and hence $\pi_{wZ}(w') = z = \pi_{wZ}(w'')$.

The remaining cases are $w = w_c$; $w \in \mathrm{Sub}(w') \smallsetminus \mathrm{Sub}(w'')$ and $w \in \mathrm{Sub}(w'') \smallsetminus \mathrm{Sub}(w')$. We claim that those cases are the only ones for which $\pi_{wZ}(w') \neq \pi_{wZ}(w'')$. In fact, we can list all the subwords of w' and w'' from their decomposition, and we have :

— for $w = w_c$,

$$\pi_{wZ}(w') = d' \neq d'' = \pi_{wZ}(w'');$$

— for $w = w_c d' u_1 ... u_i$, $i = 0, ..., n-1$,

$$\pi_{wZ}(w') = u_{i+1} \neq z_0 = \pi_{wZ}(w'');$$

— for $w = w_c d'' v_1 ... v_i$, $i = 0, ..., m-1$,

$$\pi_{wZ}(w') = z_0 \neq v_{i+1} = \pi_{wZ}(w''),$$

The previous list corresponds exactly to $\{w_c\} \cup (\mathrm{Sub}^*(w') \triangle \mathrm{Sub}^*(w''))$.

□

From the previous lemma, we can define a metric on $X * Y$. This metric counts the number of left-cosets "between" two words in $X * Y$:

Definition 1.6.11. *We call* left-cosets metric *on $X * Y$ the metric ρ_{lc} defined by, for $w', w'' \in X * Y$:*

$$\rho_{lc}(w', w'') = \#\{w \mid \pi_{wZ}(w') \neq \pi_{wZ}(w'')\}.$$

The next proposition says that the ρ_{lc}-distance between an alternating word and the empty-word e is actually the "length" of this word i.e. its number of letter :

Proposition 1.6.12. *Let $w \in X * Y$, $w \neq e$. Then :*

$$\{w' \mid \pi_{w'Z}(w) \neq \pi_{w'Z}(e)\} = Sub^*(w).$$

In particular, if $w = w_1 ... w_n$, we have $\rho_{lc}(w, e) = n$.

Proof. It is an immediate consequence of Lemma 1.6.10. □

Remark 1.6.13. *As we shall see in Chapter 2, Part 2., there is a natural notion of metric on $X * Y$ induced by two given metrics on X and Y. In this context, the metric of left-coset on $X * Y$ is exactly the one induced by the discrete metrics on X and Y.*

1.6.2 Natural space with labelled partitions on the free product

In this part, we construct a natural structure of space with labelled partitions on the free product of two spaces, given structures of space with labelled partitions on each factors.

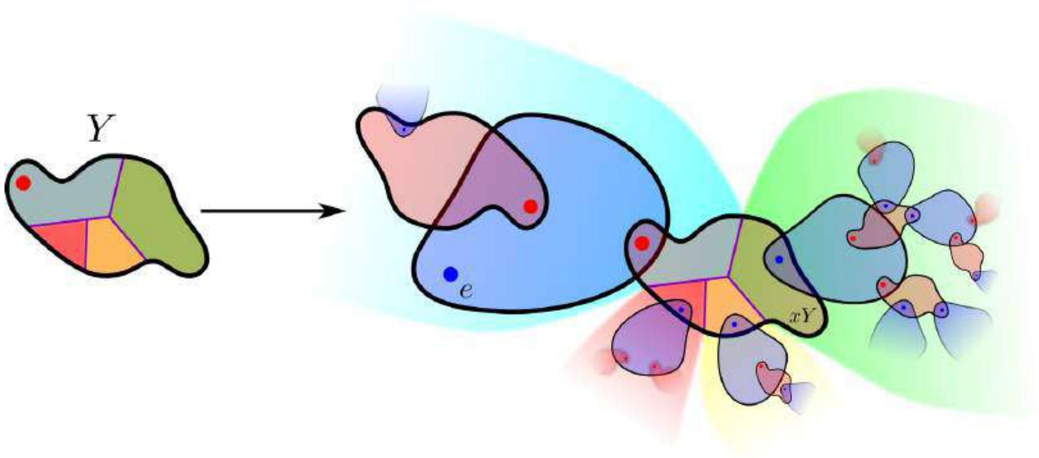
Definition 1.6.14. *Let $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ and $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ be non-empty spaces with labelled partitions and $x_0 \in X$, $y_0 \in Y$ be basepoints. Subsequently, Z will stand for X or Y as appropriate.*
Let $w \in X * Y$ and wZ the left-Z-coset associated with w (if $w = e$, it can be either eX or eY). We set, for $p_Z \in \mathcal{P}_Z$, the following labelling function on $X * Y$:

$$p_Z^{*wZ} = p_Z \circ \pi_{wZ},$$

where π_{wZ} is the wZ-projection, and we denote $\mathcal{P}_Z^{*wZ} = \{p_Z^{*wZ} \mid p_Z \in \mathcal{P}_Z\}$.
The set :

$$\mathcal{P}_{X*Y} = \bigcup_{w \in E_Y} \mathcal{P}_X^{*wX} \cup \bigcup_{w \in E_X} \mathcal{P}_Y^{*wY} \cup \mathcal{P}_X^{*eX} \cup \mathcal{P}_Y^{*eY}$$

is called the natural family of labelling functions of the free product $X * Y$ *(associated with $\mathcal{P}_X$ and $\mathcal{P}_Y$).*



Partition of $X * Y$ induced by a partition of Y via the xY-projection.

Definition 1.6.15. *Let $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ and $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ be non-empty countable spaces with labelled partitions and $x_0 \in X$, $y_0 \in Y$ be basepoints. Subsequently, Z will stand for X or Y as appropriate.*
Let $w \in X * Y$ and wZ the left-Z-coset associated with w.
For $\xi \in F_Z(\mathcal{P}_Z)$, we denote $\xi^{*wZ} : \mathcal{P}_{X*Y} \to \mathbb{K}$ the function :

$$\xi^{*wZ}(p) = \begin{cases} \xi(p_Z) \text{ if } p = p_Z^{*wZ} \in \mathcal{P}_Z^{*wZ} \\ 0 \quad \text{otherwise.} \end{cases}$$

Let $q \geq 1$. We denote $F_q(\mathcal{P}_{X*Y})$ the closure of

$$E_q(\mathcal{P}_{X*Y}) := \left\{ \sum_{wZ \text{ left-coset}} \xi_{wZ}^{*wZ} \mid \xi_{wZ} \in F_Z(\mathcal{P}_Z) \text{ with } \xi_{wZ} = 0 \text{ for all but finitely many left-cosets } wZ \right\},$$

endowed with the norm $\|.\|_{N_q}$ defined by, for $\xi = \sum_{wZ} \xi_{wZ}^{*wZ}$:

$$\|\xi\|_{N_q} := \left(\sum_{wZ} \|\xi_{wZ}\|^q_{F_Z(\mathcal{P}_Z)} \right)^{\frac{1}{q}}.$$

The vector space $F_q(\mathcal{P}_{X*Y})$ is called the q-space of functions on $\mathcal{P}_{X*Y}$ of $X * Y$.

Proposition 1.6.16 (Labelled partitions structure on a free product)**.**
Let $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ and $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ be non-empty countable spaces with labelled

partitions and $x_0 \in X$, $y_0 \in Y$ be basepoints. Consider $X * Y$ together with its natural family $\mathcal{P}_{X*Y}$ of labelling functions of the free product.

Let $q \geq 1$ and $F_q(\mathcal{P}_{X*Y})$ be the q-space of functions on $\mathcal{P}_{X*Y}$ of $X * Y$. Then, the triple $(X * Y, \mathcal{P}_{X*Y}, F_q(\mathcal{P}_{X*Y}))$ is a space with labelled partitions.

Proof. In this proof, Z will stand for X or Y, and z_0 for x_0 or y_0 as appropriate. We denote by c_Z the separation map of Z associated with $\mathcal{P}_Z$ and by c_{X*Y} the separation map associated with $\mathcal{P}_{X*Y}$.

Let $w, w', w'' \in X * Y$. For $p_Z^{*wZ} \in \mathcal{P}_Z^{*wZ}$, we have :

$$c_{X*Y}(w', w'')(p_Z^{*wZ}) = p_Z(\pi_{wZ}(w')) - p_Z(\pi_{wZ}(w'')) = c_Z(\pi_{wZ}(w'), \pi_{wZ}(w''))(p_Z).$$

It follows that $c_{X*Y}(w', w'') = \sum_{wZ} c_Z(\pi_{wZ}(w'), \pi_{wZ}(w''))^{*wZ}$ which is a finite sum since $\pi_{wZ}(w') = z_0 = \pi_{wZ}(w'')$ for all but finitely many left-cosets by Lemma 1.6.9. Thus, $c_{X*Y}(w', w'')$ belongs to $F_q(\mathcal{P}_{X*Y})$ and hence, $(X * Y, \mathcal{P}_{X*Y}, F_q(\mathcal{P}_{X*Y}))$ is a space with labelled partitions.

□

Definition 1.6.17. *Let $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ and $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ be non-empty countable spaces with labelled partitions, $x_0 \in X$, $y_0 \in Y$ be basepoints and let $q \geq 1$. Consider $X * Y$ together with its natural family $\mathcal{P}_{X*Y}$ of labelling functions on the free product and let $F_q(\mathcal{P}_{X*Y})$ be the q-space of functions on $\mathcal{P}_{X*Y}$ of $X * Y$.*

The triple $(X * Y, \mathcal{P}_{X*Y}, F_q(\mathcal{P}_{X*Y}))$ is called the natural space with labelled partitions *on the free product $X * Y$.*

1.6.3 Action on a free product of spaces with labelled partitions

1. Natural action on a free product of spaces

Let G, H be groups and X, Y be non empty sets with basepoints x_0 and y_0. Assume G acts on X via α_G and H acts on Y via α_H.

We want to define an action of the free product of groups $G * H$ on the free products of spaces $X * Y$. First, we extend the actions α_G and α_H on $X * Y$ in the following way :

In all the definitions of this part, we will observe the following rule : "*if a x_0 or a y_0 appears at the beginning of a word of $X * Y$, consider there is no letter at this place.*"

Definition 1.6.18.
We define the map $\alpha_G^* : G \to Bij(X * Y)$ *by, for* $g \in G$ *and* $w = w_1 ... w_n \in X * Y$ *:*

1) if $w_1 \in Y \setminus \{y_0\}$,

$$\alpha_G^*(g)w = (\alpha_G(g)x_0)w_1...w_n;$$

2) if $w_1 \in X \setminus \{x_0\}$,

$$\alpha_G^*(g)w = (\alpha_G(g)w_1)w_2...w_n;$$

3) if $w = e$,

$$\alpha_G^*(g)e = \alpha_G(g)x_0,$$

where $\alpha_G(g)x_0$ *is viewed as a word of at most one letter in* $X * Y$.

We define the map $\alpha_H^* : H \to Bij(X * Y)$ *by, for* $h \in H$ *and* $w = w_1 ... w_n \in X * Y$ *:*

1) if $w_1 \in X \setminus \{x_0\}$,

$$\alpha_H^*(h)w = (\alpha_H(h)y_0)w_1...w_n;$$

2) if $w_1 \in Y \setminus \{y_0\}$,

$$\alpha_H^*(h)w = (\alpha_H(h)w_1)w_2...w_n;$$

3) if $w = e$,

$$\alpha_H^*(h)e = \alpha_H(h)y_0,$$

where $\alpha_H(h)y_0$ *is viewed as a word of at most one letter in* $X * Y$.

The maps α_G^* *and* α_H^* *are called the* free extension of, respectively, α_G and α_H *on* $X * Y$.

Proposition 1.6.19. *The free extensions* α_G^* *and* α_H^* *are actions of, respectively,* G *and* H *on* $X * Y$.

Proof. For $w = w_1...w_n \in X * Y$, we clearly have $\alpha_G^*(e_G)w = w$ and, for $g_1, g_2 \in G$:

— if $w_1 \in X \setminus \{x_0\}$,

$$\alpha_G^*(g_1)(\alpha_G^*(g_2)w) = \alpha_G^*(g_1)((\alpha_G(g_2)w_1)w_2...w_n) = (\alpha_G(g_1g_2)w_1)w_2...w_n = \alpha_G^*(g_1g_2)w;$$

— if $w_1 \in Y \setminus \{y_0\}$,

$$\alpha_G^*(g_1)(\alpha_G^*(g_2)w) = \alpha_G^*(g_1)((\alpha_G(g_2)x_0)w_1...w_n) = (\alpha_G(g_1g_2)x_0)w_1...w_n = \alpha_G^*(g_1g_2)w;$$

— if $w = e$,

$$\alpha_G^*(g_1)(\alpha_G^*(g_2)e) = \alpha_G^*(g_1)((\alpha_G(g_2)x_0) = \alpha_G(g_1g_2)x_0 = \alpha_G^*(g_1g_2)e.$$

It follows that α_G^* is an action of G on $X * Y$. Similar arguments hold for α_H^*. $\square$

Lemma 1.6.20. *Let G_1, G_2 be groups and E be a set. Assume there exists actions α_1, α_2 of, respectively, G_1 and G_2 on E. Then the map $\alpha : G_1 * G_2 \to Bij(E)$ such that, for $\gamma = \gamma_1 ... \gamma_n \in G_1 * G_2$ and $\varepsilon \in E$:*

$$\alpha(\gamma)\varepsilon := \alpha_{1,2}(\gamma_1) \circ \alpha_{1,2}(\gamma_2) \circ ... \circ \alpha_{1,2}(\gamma_n)(\varepsilon),$$

defines an action of $G_1 * G_2$ on E where $\alpha_{1,2}$ stands for α_1 or α_2 as appropriate.

Proof. Let $\Gamma = G_1 * G_2$ and e_Γ be the empty word of Γ.
We have, for $\varepsilon \in E$, $\alpha(e_\Gamma)\varepsilon = id_E(\varepsilon) = \varepsilon$ and, for $\gamma = \gamma_1 ... \gamma_n, \gamma' = \gamma'_1 ... \gamma'_m \in \Gamma$:

$$\begin{aligned} \alpha(\gamma)(\alpha(\gamma')\varepsilon) &= \alpha(\gamma)(\alpha_{1,2}(\gamma'_1) \circ ... \circ \alpha_{1,2}(\gamma'_m)(\varepsilon)), \\ &= \alpha_{1,2}(\gamma_1) \circ ... \circ \alpha_{1,2}(\gamma_n) \circ \alpha_{1,2}(\gamma'_1) \circ ... \circ \alpha_{1,2}(\gamma'_m)(\varepsilon), \\ \alpha(\gamma)(\alpha(\gamma')\varepsilon) &= \alpha(\gamma\gamma')(\varepsilon). \end{aligned}$$

Hence, α is an action of Γ on E. $\square$

Definition 1.6.21. *We denote by α^* the action of $G * H$ on $X * Y$ given by Lemma 1.6.20 from the free extension actions α_G^* and α_H^* and we call α^* the* natural action of $G * H$ on *$X * Y$ (associated with α_G and α_H).*

Remark 1.6.22. *The terminology "natural" comes from the fact that if α_G and α_H are respectively the actions of G and H by left-translation on themselves, then α^* is the action of $G * H$ by translation on itself.*

Lemma 1.6.23. *Let Z be X or Y as appropriate. We have :*

— *for $g \in G$ and wZ a left-coset,*

$$\pi_{wZ} \circ \alpha_G^*(g) = \begin{cases} \pi_{\alpha_G^*(g^{-1})wZ} & \text{if } wZ \neq eX, \\ \alpha_G(g) \circ \pi_{eX} & \text{if } wZ = eX. \end{cases}$$

— *for $h \in H$ and wZ a left-coset,*

$$\pi_{wZ} \circ \alpha_H^*(h) = \begin{cases} \pi_{\alpha_H^*(h^{-1})wZ} & \text{if } wZ \neq eY, \\ \alpha_H(h) \circ \pi_{eY} & \text{if } wZ = eY. \end{cases}$$

Proof. Let $g \in G$.

First case : $w \neq e$. Let $w' \in X * Y$. Notice that $w \in \mathrm{Sub}(\alpha_G^*(g)w')$ if, and only if, $\alpha_G^*(g)w' = wzu$ with $z \in Z$ (possibly z_0) and $u \in X * Y$ ($u = e$ if $z = z_0$). Then we have, $w \in \mathrm{Sub}(\alpha_G^*(g)w')$ if, and only if,

$$w' = \alpha_G^*(g^{-1})(wzu) = \alpha_G^*(g^{-1})(w)zu \text{ since } w \neq e.$$

Hence, for all $w' \in X * Y$, $\pi_{wZ} \circ \alpha_G^*(g)(w') = \pi_{\alpha_G^*(g^{-1})wZ}(w')$.

Second case : $wZ = eY$. Notice that $(\alpha_G^*(g^{-1})e)Y = (\alpha_G(g^{-1})x_0)Y$ if $\alpha_G(g^{-1})x_0 \neq x_0$ and $(\alpha_G^*(g^{-1})e)Y = eY$ if $\alpha_G(g^{-1})x_0 = x_0$. Let $w' \in X * Y$.
If $w' = e$, $\pi_{eY} \circ \alpha_G^*(g)(w') = y_0 = \pi_{\alpha_G^*(g^{-1})eY}(w')$.
If $w' = yu$ with $y \in Y \setminus \{y_0\}$, $u \in X * Y$, then :

$$\pi_{\alpha_G^*(g^{-1})eY}(w') = \begin{cases} y_0 & \text{if } \alpha_G(g^{-1})(x_0) \neq x_0, \\ y & \text{if } \alpha_G(g^{-1})(x_0) = x_0. \end{cases}$$

Now, we have $\alpha_G^*(g)w' = \alpha_G(g)(x_0)yu$, then :

$$\pi_{eY} \circ \alpha_G^*(g)(w') = \begin{cases} y_0 & \text{if } \alpha_G(g)(x_0) \neq x_0, \\ y & \text{if } \alpha_G(g)(x_0) = x_0. \end{cases}$$

And if $w' = xu$ with $x \in X \setminus \{y_0\}$, $u = u_1 ... u_n \in X * Y$ with $u_1 \in Y$ (if $u = e$ we set by convention $u_1 = y_0$), then :

$$\pi_{\alpha_G^*(g^{-1})eY}(w') = \begin{cases} y_0 & \text{if } \alpha_G(g^{-1})(x_0) \neq x, \\ u_1 & \text{if } \alpha_G(g^{-1})(x_0) = x. \end{cases}$$

And we have $\alpha_G^*(g)w' = \alpha_G(g)(x)u$, then :

$$\pi_{eY} \circ \alpha_G^*(g)(w') = \begin{cases} y_0 & \text{if } \alpha_G(g)(x) \neq x_0, \\ u_1 & \text{if } \alpha_G(g)(x) = x_0. \end{cases}$$

It follows that in this case, for every $w' \in X * Y$, $\pi_{eY} \circ \alpha_G^*(g)(w') = \pi_{\alpha_G^*(g^{-1})eY}(w')$.

Third case : $wZ = eX$.

If $w' = e$, $\pi_{eX} \circ \alpha_G^*(g)(w') = \alpha_G(g)x_0 = \alpha_G(g) \circ \pi_{eX}(w')$.
If $w' = yu$ with $y \in Y \smallsetminus \{y_0\}$, $u \in X * Y$, we have :

$$\pi_{eX} \circ \alpha_G^*(g)(w') = \alpha_G(g)x_0 = \alpha_G(g) \circ \pi_{eX}(w').$$

And if $w' = xu$ with $x \in X \smallsetminus \{x_0\}$, $u \in X * Y$, we have :

$$\pi_{eX} \circ \alpha_G^*(g)(w') = \alpha_G(g)x = \alpha_G(g) \circ \pi_{eX}(w').$$

Then, for every $w' \in X * Y$, $\pi_{eX} \circ \alpha_G^*(g)(w') = \alpha_G(g) \circ \pi_{eX}(w')$.

This ends the proof of the first statement, and by permuting the role of G and H, and X and Y in the previous arguments, the second statement holds as well. $\square$

Proposition 1.6.24. *Let $\gamma \in G * H$. Set $Z = X$ or Y and $K = G$ or H in such a way that K acts on Z. Then, for every left-coset wZ, there exists $k_{w,\gamma} \in K$ and a unique $\gamma_{w,\gamma} \in G * H$ such that :*

$$\pi_{wZ} \circ \alpha^*(\gamma) = \alpha_K(k_{w,\gamma}) \circ \pi_{\alpha^*(\gamma_{w,\gamma})wZ}.$$

Moreover, the map : $wZ \mapsto \alpha^*(\gamma_{w,\gamma})wZ$ is a bijection on the set of left-cosets.

Proof. Let $\gamma = \gamma_1...\gamma_n$.
We prove the first statement by induction on the word length l of the subword $\gamma_{(l)} = \gamma_1...\gamma_l$ of γ. By Lemma 1.6.20, the case $l = 1$ is true.
Let $1 \leq l < n$ and assume that for every left-coset wZ, there exists $k_{w,\gamma_{(l)}} \in K$ and a unique $\gamma_{w,\gamma_{(l)}} \in G * H$ such that $\pi_{wZ} \circ \alpha^*(\gamma_{(l)}) = \alpha_K(k_{w,\gamma_{(l)}}) \circ \pi_{\alpha^*(\gamma_{w,\gamma_{(l)}})wZ}$.
We can assume, up to permute G and H and X and Y, that $\gamma_l \in H \smallsetminus \{e_H\}$; it follows that $\gamma_{l+1} \in G \smallsetminus \{e_G\}$. Let wZ be a left-coset.
If $wZ = wX$ is a left-X-coset, then, $k_{w,\gamma_{(l)}} \in G$ and :

$$\pi_{wX} \circ \alpha^*(\gamma_{(l+1)}) = \pi_{wX} \circ \alpha^*(\gamma_{(l)}) \circ \alpha_G^*(\gamma_{l+1}) = \alpha_G(k_{w,\gamma_{(l)}}) \circ \pi_{\alpha^*(\gamma_{w,\gamma_{(l)}})wX} \circ \alpha_G^*(\gamma_{l+1}).$$

But, by Lemma 1.6.20 :

$$\pi_{\alpha^*(\gamma_{w,\gamma_{(l)}})wX} \circ \alpha_G^*(\gamma_{l+1}) = \begin{cases} \pi_{\alpha_G^*(\gamma_{l+1}^{-1})\alpha^*(\gamma_{w,\gamma_{(l)}})wX} & \text{if } \alpha^*(\gamma_{w,\gamma_{(l)}})wX \neq eX, \\ \alpha_G(\gamma_{l+1}) \circ \pi_{eX} & \text{if } \alpha^*(\gamma_{w,\gamma_{(l)}})wX = eX. \end{cases}$$

and hence,

$$\pi_{wX} \circ \alpha^*(\gamma_{(l+1)}) = \begin{cases} \alpha_G(k_{w,\gamma_{(l)}}) \circ \pi_{\alpha^*(\gamma_{l+1}^{-1}\gamma_{w,\gamma_{(l)}})wX} & \text{if } \alpha^*(\gamma_{w,\gamma_{(l)}})wX \neq eX, \\ \alpha_G(k_{w,\gamma_{(l)}}\gamma_{l+1}) \circ \pi_{eX} & \text{if } \alpha^*(\gamma_{w,\gamma_{(l)}})wX = eX. \end{cases}$$

And, If $wZ = wY$ is a left-Y-coset, then, $k_{w,\gamma_{(l)}} \in H$ and :

$$\pi_{wY} \circ \alpha^*(\gamma_{(l+1)}) = \pi_{wY} \circ \alpha^*(\gamma_{(l)}) \circ \alpha_G^*(\gamma_{l+1}) = \alpha_H(k_{w,\gamma_{(l)}}) \circ \pi_{\alpha^*(\gamma_{w,\gamma_{(l)}})wY} \circ \alpha_G^*(\gamma_{l+1}).$$

Again, by Lemma 1.6.20, we have :

$$\pi_{\alpha^*(\gamma_{w,\gamma_{(l)}})wY} \circ \alpha_G^*(\gamma_{l+1}) = \pi_{\alpha_G^*(\gamma_{l+1}^{-1})\alpha^*(\gamma_{w,\gamma_{(l)}})wX},$$

since $\alpha^*(\gamma_{w,\gamma_{(l)}})wY \neq eX$; and hence,

$$\pi_{wY} \circ \alpha^*(\gamma_{(l+1)}) = \alpha_H(k_{w,\gamma_{(l)}}) \circ \pi_{\alpha^*(\gamma_{l+1}^{-1}\gamma_{w,\gamma_{(l)}})wX}.$$

As a conclusion, we have :

1) for $wZ = wX$ with $\alpha^*(\gamma_{w,\gamma_{(l)}})w \neq e$,

$$k_{w,\gamma_{(l+1)}} = k_{w,\gamma_{(l)}} \text{ and } \gamma_{w,\gamma_{(l+1)}} = \gamma_{l+1}^{-1}\gamma_{w,\gamma_{(l)}},$$

2) for $wZ = wX$ with $\alpha^*(\gamma_{w,\gamma_{(l)}})w = e$,

$$k_{w,\gamma_{(l+1)}} = k_{w,\gamma_{(l)}}\gamma_{l+1} \text{ and } \gamma_{w,\gamma_{(l+1)}} = \gamma_{w,\gamma_{(l)}},$$

3) for $wZ = wY$,

$$k_{w,\gamma_{(l+1)}} = k_{w,\gamma_{(l)}} \text{ and } \gamma_{w,\gamma_{(l+1)}} = \gamma_{l+1}^{-1}\gamma_{w,\gamma_{(l)}}.$$

In each case, $\gamma_{w,\gamma_{(l+1)}}$ is unique by uniqueness of $\gamma_{w,\gamma_{(l)}}$. This ends the proof by induction.

Let us show that $wZ \mapsto \alpha^*(\gamma_{w,\gamma})wZ$ is a bijection on the set of left-cosets. More precisely, we show that $\phi_{\gamma,X} : wX \mapsto \alpha^*(\gamma_{w,\gamma})wX$ is a bijection on the set of left-X-cosets and that $\phi_{\gamma,Y} : wY \mapsto \alpha^*(\gamma_{w,\gamma})wY$ is a bijection on the set of left-Y-cosets. First, remark that two left-cosets wZ and $w'Z'$ are equal if, and only if $Z = Z'$ and $w = w'$.
We proceed again by induction on the word length l of the subword $\gamma_{(l)} = \gamma_1...\gamma_l$ of γ.

For $l = 1$, if $\gamma_1 \in G$, notice that for $w \in E_Y$, $\alpha^*(\gamma_1^{-1})w \neq e \neq \alpha^*(\gamma_1)w$ and hence :

$$\phi_{\gamma_1,X}(wX) = \begin{cases} \alpha^*(\gamma_1^{-1})wX & \text{if } wX \neq eX, \\ eX & \text{if } wX = eX. \end{cases}$$

clearly defines a bijection on the set of left-X-cosets. And if $\gamma_1 \in H$, we have, for all wX, $\phi_{\gamma_1,X}(wX) = \alpha^*(\gamma_1^{-1})wX$ which again clearly defines a bijection.
Let $1 \leq l < n$ and assume $\phi_{\gamma_{(l)},X}$ is a bijection on left-X-cosets. We assume, up to permute G and H, that $\gamma_{l+1} \in G \smallsetminus \{e_G\}$. We have, using 1) and 2) :

$$\phi_{\gamma_{(l+1)},X}(wX) = \begin{cases} \alpha^*(\gamma_{l+1}^{-1}\gamma_{w,\gamma_{(l)}})wX & \text{if } \alpha^*(\gamma_{w,\gamma_{(l)}})wX \neq eX, \\ eX & \text{if } \alpha^*(\gamma_{w,\gamma_{(l)}})wX = eX. \end{cases}$$

which is a bijection, since $wX \mapsto \alpha^*(\gamma_{w,\gamma_{(l)}})wX$ is a bijection. This ends the induction and it follows that $\phi_{\gamma,X}$ is a bijection on the set of left-X-cosets. By a similar induction, we conclude that $\phi_{\gamma,Y}$ is a bijection on left-Y-cosets. □

2. Action by automorphisms on a free product of spaces with labelled partitions

Proposition 1.6.25.
Let $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ and $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ be non-empty countable spaces with labelled partitions and $x_0 \in X$, $y_0 \in Y$ be basepoints. Let G and H be discrete countable groups acting by automorphisms on $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ and $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ respectively.

Let $q \geq 1$. Then $G * H$ acts by automorphisms on the natural space with labelled partitions on the free product $(X * Y, \mathcal{P}_{X*Y}, F_q(\mathcal{P}_{X*Y}))$ via the natural action of $G * H$ on $X * Y$.

Proof. Subsequently, Z will stand for X or Y, z_0 for x_0 or y_0 and K for G or H as appropriate.
We Denote :
- τ_G the G-action of G on X and for $g \in G$, $p \in \mathcal{P}_X$, $\Phi_{\tau_G(g)}(p) := p \circ \tau_G(g)$;
- τ_H the H-action of H on Y and for $h \in H$, $p \in \mathcal{P}_Y$, $\Phi_{\tau_H(h)}(p) := p \circ \tau_H(h)$;
- τ^* the the natural action of $G * H$ on $X * Y$ and for $\gamma \in \Gamma$, $p \in \mathcal{P}_{X*Y}$, $\Phi_{\tau^*(\gamma)}(p) := p \circ \tau^*(\gamma)$.

Let $p \in \mathcal{P}_{X*Y}$ and $\gamma = \gamma_1 ... \gamma_n \in G * H$. Then there exists $p_Z \in \mathcal{P}_Z$ and a left-coset wZ such that $p = p_Z^{*wZ}$, and we have $\Phi_{\tau^*(\gamma)}(p_Z^{*wZ}) = p_Z \circ \pi_{wZ} \circ \tau^*(\gamma)$. Hence, by Proposition

1.6.24, there exists $k_w \in K$ and $\gamma_w \in G * H$ such that :

$$\Phi_{\tau^*(\gamma)}(p_Z^{*wZ}) = p_Z \circ \tau_K(k_w) \circ \pi_{\tau^*(\gamma_w)wZ} = (p_Z \circ \tau_K(k_w))^{*\tau^*(\gamma_w)wZ}.$$

It follows that $\Phi_{\tau^*(\gamma)}(p_Z^{*wZ})$ belongs to P_{X*Y} since $p_Z \circ \tau_K(k_w) \in P_Z$.

For $\xi = \sum_{wZ} \xi_{wZ}^{*wZ} \in E_q(\mathcal{P}_{X*Y})$, we have :

$$\begin{aligned} \xi \circ \Phi_{\tau^*(\gamma)}(p) &= \xi((p_Z \circ \tau_K(k_w))^{*\tau^*(\gamma_w)wZ}), \\ &= \xi_{\tau^*(\gamma_w)wZ}(p_Z \circ \tau_K(k_w)), \\ \xi \circ \Phi_{\tau^*(\gamma)}(p) &= (\xi_{\tau^*(\gamma_w)wZ} \circ \Phi_{\tau_K(k_w)})^{*wZ}(p). \end{aligned}$$

Thus,

$$\xi \circ \Phi_{\tau^*(\gamma)} = \sum_{wZ} (\xi_{\tau^*(\gamma_w)wZ} \circ \Phi_{\tau_K(k_w)})^{*wZ},$$

and then, by making the substitution $wZ = \tau^*(\gamma_w)wZ$ which is bijective by Proposition 1.6.24, we have :

$$\xi \circ \Phi_{\tau^*(\gamma)} = \sum_{wZ} (\xi_{wZ} \circ \Phi_{\tau_K(k_w)})^{*\tau^*(\gamma_w^{-1})wZ} \in F_q(\mathcal{P}_{X*Y}),$$

since $\xi_{wZ} = 0$ for all but finitely many wZ.
By completeness of $F_q(\mathcal{P}_{X*Y})$, for all $\xi \in F_q(\mathcal{P}_{X*Y})$, $\xi \circ \Phi_{\tau^*(\gamma)} \in F_q(\mathcal{P}_{X*Y})$.
Moreover, for $\xi = \sum_{wZ} \xi_{wZ}^{*wZ} \in E_q(\mathcal{P}_{X*Y})$, we have :

$$\begin{aligned} \|\xi \circ \Phi_{\tau^*(\gamma)}\|_{N_q}^q &= \textstyle\sum_{wZ} \|\xi_{wZ} \circ \Phi_{\tau_K(k_w)}\|_{F_Z(\mathcal{P}_Z)}^q \\ &= \textstyle\sum_{wZ} \|\xi_{wZ}\|_{F_Z(\mathcal{P}_Z)}^q, \\ \|\xi \circ \Phi_{\tau^*(\gamma)}\|_{N_q}^q &= \|\xi\|_{N_q}^q, \end{aligned}$$

since, for all wZ, $\|\xi_{wZ} \circ \Phi_{\tau_K(k_w)}\|_{F_Z(\mathcal{P}_Z)} = \|\xi_{wZ}\|_{F_Z(\mathcal{P}_Z)}$.
Thus, by density of $E_q(\mathcal{P}_{X*Y})$ in $F_q(\mathcal{P}_{X*Y})$, for all $\xi \in F_q(\mathcal{P}_{X*Y})$, $\|\xi \circ \Phi_{\tau^*(\gamma)}\|_{N_q} = \|\xi\|_{N_q}$.
It follows that $G * H$ acts by automorphisms on $(X * Y, \mathcal{P}_{X*Y}, F_q(\mathcal{P}_{X*Y}))$.

□

In the free product $X * Y$ where X and Y are countable, there is two ways of leaving finite sets for a word $w = w_1 ... w_n \in X * Y$:

— leaving finite sets in a particular left-coset i.e. there exists j such that w_j leaves every finite set in Z ;

— the number of letters n of w is going to infinity i.e. $\rho_{lc}(w,e) \to +\infty$.

Even if G, H acts properly by automorphisms on their respective spaces with labelled partitions, the action of $G * H$ on the natural space with labelled partitions of the free product need not be proper since those hypothesis only implies the properness for words that leave every finite set in some given left-cosets. We have to add another structure of labelled partitions on which $G * H$ acts that ensures the properness of the action when $\rho_{lc}(w,e) \to +\infty$. That is the point of the following definition :

Definition 1.6.26. *Let $q \geq 1$, X, Y be non-empty countable sets, x_0, y_0 be basepoints and $(X, \varDelta_X, \ell^q(\varDelta_X))$,$(Y, \varDelta_Y, \ell^q(\varDelta_Y))$ be the q-naive spaces with labelled partitions on, respectively, X and Y.*
We denote by $(X * Y, \varDelta_{X*Y}, F_q(\varDelta_{X*Y}))$ the natural space with labelled partitions on the free product of $(X, \varDelta_X, \ell^q(\varDelta_X))$ and $(Y, \varDelta_Y, \ell^q(\varDelta_Y))$.

Lemma 1.6.27. *Let $q \geq 1$, X, Y be non-empty countable sets, x_0, y_0 be basepoints and $(X, \varDelta_X, \ell^q(\varDelta_X))$,$(Y, \varDelta_Y, \ell^q(\varDelta_Y))$ be the q-naive spaces with labelled partitions on, respectively, X and Y. Let $(X * Y, \varDelta_{X*Y}, F_q(\varDelta_{X*Y}))$ be the natural space with labelled partitions on the free product of X and Y. Then,*

$$\|c_{\varDelta_{X*Y}}(w,e)\|_{N_q} \underset{\rho_{lc}(w,e)\to+\infty}{\longrightarrow} +\infty,$$

where $c_{\varDelta_{X*Y}}$ is the separation map on $X * Y$ associated with $\varDelta_{X*Y}$.

Proof. Let $w \in X * Y$. By Proposition 1.6.12 we have $\{w' \mid \pi_{w'Z}(w) \neq z_0\} = \mathrm{Sub}^*(w)$, and then :

$$\begin{aligned}
\|c_{\varDelta_{X*Y}}(w,e)\|_{N_q} &= \sum_{w'Z}\sum_{z\in Z} |c_{\varDelta_{X*Y}}(w,e)(\varDelta_z^{*w'Z})|^q \\
&= \sum_{w'\in \mathrm{Sub}^*(w)} \sum_{z\in Z} \frac{1}{2}|\delta_z(\pi_{w'Z}(w)) - \delta_z(z_0)|^q \\
\|c_{\varDelta_{X*Y}}(w,e)\|_{N_q} &= \#\mathrm{Sub}^*(w) = \rho_{lc}(w,e).
\end{aligned}$$

It follows that if $\rho_{lc}(w,e) \to +\infty$, then

$$\|c_{\varDelta_{X*Y}}(w,e)\|_{N_q} \to +\infty.$$

□

Theorem 5.

Let $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ and $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ be non-empty countable spaces with labelled partitions and $x_0 \in X$, $y_0 \in Y$ be basepoints. Let G and H be discrete countable groups acting (resp. acting properly) by automorphisms on $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ and $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$ respectively such that no element of G fixes x_0 and no element of H fixes y_0.

Let $q \geq 1$. Then there exists a structure of space with labelled partitions $(M, \mathcal{P}_M, F(\mathcal{P}_M))$ on which $G * H$ acts (resp. acts properly) by automorphisms.
More precisely, $(M, \mathcal{P}_M, F(\mathcal{P}_M))$ is the natural space with labelled partitions on the direct product $M = (X * Y) \times (G * H)$ where :

- *on $X*Y$, we consider the natural space with labelled partitions $(X*Y, \mathcal{P}_{X*Y}, F_q(\mathcal{P}_{X*Y}))$ on the free product of $(X, \mathcal{P}_X, F_X(\mathcal{P}_X))$ and $(Y, \mathcal{P}_Y, F_Y(\mathcal{P}_Y))$;*
- *on $G*H$, we consider the natural space with labelled partitions $(G*H, \Delta_{G*H}, F_q(\Delta_{G*H}))$ on the free product of the q-naive spaces with labelled partitions $(G, \Delta_G, \ell^q(\Delta_G))$, $(H, \Delta_H, \ell^q(\Delta_H))$ on, respectively, G and H.*

Proof of Theorem 5. By Proposition 1.6.25, $G * H$ acts by automorphisms on both $(X * Y, \mathcal{P}_{X*Y}, F_q(\mathcal{P}_{X*Y}))$ and $(G * H, \Delta_{G*H}, F_q(\Delta_{G*H}))$. We set $M = (X * Y) \times (G * H)$ and we consider the natural space with labelled partitions $(M, \mathcal{P}_M, F_q(\mathcal{P}_M))$ on the direct product where :

$$\mathcal{P} = \mathcal{P}_{X*Y}^{\oplus_1} \cup \Delta_{G*H}^{\oplus_2},$$

and

$$F_q(\mathcal{P}) \simeq F_q(\mathcal{P}_{X*Y}) \oplus \ell^q(\Delta_{G*H}).$$

Then, by Proposition 1.4.5, $(G*H) \times (G*H)$ acts by automorphisms on $(M, \mathcal{P}_M, F_q(\mathcal{P}_M))$ via the action $(\gamma, \gamma').(w, \gamma'') = (\tau^*(\gamma)w, \gamma'\gamma'')$. Hence, $G * H$ acts by automorphisms on $(M, \mathcal{P}_M, F_q(\mathcal{P}_M))$, where $G * H$ is viewed as the diagonal subgroup $\{(\gamma, \gamma) \mid \gamma \in G * H\} < (G * H) \times (G * H)$.

It remains to prove that the $G * H$-action on $(M, \mathcal{P}_M, F_q(\mathcal{P}_M))$ is proper. Let $\gamma = \gamma_1 ... \gamma_n \in G * H$. Notice that, since x_0 is not fixed by any element in G and y_0 is not fixed by any element of H, $\tau^*(\gamma)e = (\tau_K(\gamma_1)z_0)...(\tau_K(\gamma_n)z_0)$ and then :

$$\pi_{wZ}(\tau^*(\gamma)e) = \begin{cases} \tau_K(\gamma_{i+1})z_0 & \text{if } wZ = (\tau_K(\gamma_1)z_0)...(\tau_K(\gamma_i)z_0)Z,\ i = 0, ..., n-1, \\ z_0 & \text{otherwise.} \end{cases}$$

Hence, we have :

$$\begin{aligned} \|c_{\mathcal{P}_M}(\gamma.(e, e_\Gamma), (e, e_\Gamma))\|^q_{F_q(\mathcal{P}_M)} &= \|c_{\mathcal{P}_{X*Y}}(\tau^*(\gamma)e, e)\|^q_{F_q(\mathcal{P}_{X*Y})} + \|c_{\Delta_{G*H}}(\gamma, e_\Gamma)\|^q_q \\ &= \sum_{i=1}^{n} \|c_{\mathcal{P}_Z}(\tau_K(\gamma_i)z_0, z_0)\|^q_{F_Z(\mathcal{P}_Z)} + \sum_{i=1}^{n} \|c_{\Delta_K}(\gamma_i, e_K)\|^q_q. \end{aligned}$$

If there exists j such that γ_j leaves every finite set in K, then, since the K-action on Z is proper, we have :

$$\|c_{\mathcal{P}_M}(\gamma.(e, e_\Gamma), (e, e_\Gamma))\|_{F_q(\mathcal{P}_M)} \geq \|c_{\mathcal{P}_Z}(\tau_K(\gamma_j)z_0, z_0)\|_{F_Z(\mathcal{P}_Z)} \to +\infty,$$

and if $n = \rho_{lc}(\gamma, e) \to +\infty$, by Lemma 1.6.27, we have :

$$\|c_{\mathcal{P}_M}(\gamma.(e, e_\Gamma), (e, e_\Gamma))\|^q_{F_q(\mathcal{P}_M)} \geq \sum_{i=1}^{n} \|c_{\Delta_K}(\gamma_i, e_K)\|^q_q = n \to +\infty.$$

Hence, $G * H$ acts properly on $(M, \mathcal{P}_M, F_q(\mathcal{P}_M))$. $\square$

Corollary 6.
Let $p \geq 1$ with $p \notin 2\mathbb{Z} \setminus \{2\}$ and G, H be discrete countable groups. G and H have property PL^p if, and only if, $G * H$ has property PL^p.

Proof of Corollary 6. Assume that G, H have property Pl^p. By Corollary 2, there exists structures of spaces with labelled partitions $(G, \mathcal{P}_G, F_G(\mathcal{P}_G))$ and $(H, \mathcal{P}_H, F_H(\mathcal{P}_H))$ on which G and H respectively, act properly by automorphisms by left-translation on itself. Moreover $F_G(\mathcal{P}_G)$ is isometrically isomorph to a closed subspace of a L^p space and so does $F_H(\mathcal{P}_H)$. Remark that no element of G fixes the identity element e_G of G and no element of H fixes e_H in H. Thus we can apply Theorem 5 to $G \curvearrowright (G, \mathcal{P}_G, F_G(\mathcal{P}_G))$ and $H \curvearrowright (H, \mathcal{P}_H, F_H(\mathcal{P}_H))$ and then there exists a space with labelled partitions $(M, \mathcal{P}_M, F_M(\mathcal{P}_M))$ on which $G * H$ acts properly by automorphisms where

$$F_M(\mathcal{P}_M) \simeq \bigoplus_{\text{countable}}^{p} F_G(\mathcal{P}_G) \oplus \bigoplus_{\text{countable}}^{p} F_H(\mathcal{P}_H) \oplus F_p(\Delta_{G*H}).$$

Hence $F_M(\mathcal{P}_M)$ is isometrically isomorph to a closed subspace of a L^p space by Proposition 1.2.12. By Corollary 2, it follows that $G * H$ has property PL^p.

The necessary condition is clear since property PL^p is stable by taking closed subgroups and $G < G * H$ and $H < G * H$. $\square$

Chapter 2

0n δ-median spaces

2.1 Introduction

A median algebra is a set X together with a map $\mu : X \times X \times X \to X$ which satisfies the following conditions :

1. $\mu(x, x, y) = x$,
2. $\mu(x, y, z) = \mu(y, z, x) = \mu(x, z, y)$
3. $\mu(\mu(x, y, z), u, v) = \mu(x, \mu(y, u, v), \mu(z, u, v))$.

In the particular setting of metric spaces, a median space is a geodesic metric space for which, given any triple a, b, c, there exists a unique element m called median point between any two point in that triple. A point x is *between a and b* if $d(a, x) + d(x, b) = d(a, b)$. There is a significant litterature about median algebras and median spaces (see for instance [BH83], [Bas01],[Rol98],[BC08],[Bow13b]...). Recent works have been made in generalizing median spaces : Bowditch introduced in [Bow13a] the notion of coarse median geodesic metric spaces ; such spaces satisfy the axioms of median algebra up to bounded distance. In [CDH10], Chatterji, Drutu and Haglund showed the existence of a strong connection between median spaces and space with measured walls. In this latter paper, they introduced a "quasification" of median spaces, namely the δ-median spaces which on one hand generalizes median spaces, and on the other hand, contains the Gromov hyperbolic spaces as we shall see in Section 2.4. In the following review, we study a strong version of the quasi-median definition. Originally, a (weak) δ-median space, is a metric space which is L_δ, i.e. given any triple, there exists some points called δ-median points which are between, up to δ, any two points in that triple (see Definition 2.2.1) and such that the diameter of δ-median points is bounded uniformly over triples in this space. Elder in

[Eld04] showed that finitely generated groups satisfying the L_δ property are almost convex and have a sub-cubic Dehn function. Here, we consider a stronger definition where the L_δ condition is replaced by the so called L'_δ condition, namely, for any triple, there exists a δ-thin triangle in the sense of Definition 2.3.3. In this study of quasi-median spaces, we discuss the axiom of the δ-median definition and establish some basic properties of this structure. Then we show that Gromov hyperbolic spaces are quasi-median and we build an explicit quasi-median space which is neither hyperbolic nor median. Finally we explore the stability of the quasi-median condition by direct product and free product.

2.2 Preliminaries

In this section, we record for further use a few definitions and basic properties about metric spaces.
Let (X, d) be a metric space.

2.2.1 Thick intervals

Definition 2.2.1. *Let $a, b \in X$ and $\alpha \geqslant 0$.*
We call α-interval *between a and b the set :*

$$[a, b]_\alpha = \{x \in X \mid d(a, x) + d(x, b) \leqslant d(a, b) + \alpha\}.$$

Moreover, if x belongs to $[a, b]_\alpha$, we say that x is α-between *a and b.*

For simplicity, when $\alpha = 0$, we denote $[a, b] := [a, b]_0 = \{x \in X \mid d(a, x) + d(x, b) = d(a, b)\}$.

Proposition 2.2.2. *Let $\alpha, \beta \geqslant 0$ and $a, b, c \in X$. If $c \in [a, b]_\alpha$, then $[c, b]_\beta \subset [a, b]_{\alpha+\beta}$. In particular, If $c \in [a, b]$, then $[c, b] \subset [a, b]$.*

Proof. Let $x \in [c, b]_\beta$. Then we have :

$$d(a, x) + d(x, b) \leqslant d(a, c) + d(c, x) + d(x, b) \leq d(a, b) + \alpha + \beta.$$

□

Definition 2.2.3. *Let $\alpha \geq 0$. We say that x and y in X are α-close if $d(x,y) \leq \alpha$. Let Y be a subset of X. The set $V_\alpha(Y) = \{x \in X \mid \exists y \in Y,\ x, y \text{ are } \alpha\text{-close}\}$ is called the α-neighbourhood of Y in X.*

Lemma 2.2.4. *Let $\alpha, \beta \geq 0$ and $a, b \in X$. We have :*

$$V_\beta([a,b]_\alpha) \subset [a,b]_{\alpha+2\beta}.$$

In particular, we have $V_{\frac{\alpha}{2}}([a,b]) \subset [a,b]_\alpha$.

Proof. Let $x \in V_\beta([a,b]_\alpha)$. Then there exists $y \in [a,b]_\alpha$ such that $d(x,y) \leq \beta$, and hence :

$$d(a,x) + d(x,b) \leq d(a,y) + d(y,b) + 2d(x,y) \leq d(a,b) + \alpha + 2\beta.$$

□

2.2.2 Geodesics and triangles

We recall here the notion of geodesics in a metric space and we introduce the notion of triangles in geodesic metric spaces :

Definition 2.2.5. *Let (X,d) be a metric space.*

- *We call* geodesic path *from a to b a finite sequence $(x_0, ..., x_n)$ of elements of X such that, $x_0 = a$, $x_n = b$ and*

$$\sum_{i=0}^{n-1} d(x_i, x_{i+1}) = d(a,b).$$

- *We call* geodesic arc *from a to b in X an isometry $\gamma : [0, d(x,y)] \to X$ such that $\gamma(0) = x$ and $\gamma(d(x,y)) = y$.*
- *We say that (X,d) is a* geodesic *metric space if, for all $a, b \in X$, there exists a geodesic arc from a to b.*

Remark 2.2.6. *If $(x_0, ..., x_n)$ is a geodesic path from a to b with $a \neq b$, then $(x_0, ..., x_{n-1})$ is a geodesic path from a to x_{n-1} since*

$$d(a, x_{n-1}) \leq \sum_{i=0}^{n-2} d(x_i, x_{i+1}) = d(a,b) - d(x_{n-1}, b) \leq d(a, x_{n-1}).$$

It follows that, for all $i \in \{0, ..., n\}$, $(x_0, ..., x_i)$ is a geodesic path from a to x_i.

Lemma 2.2.7. *Let (X,d) be a metric space and let $a,b,c \in X$ such that (a,b,c) is a geodesic path from a to c. Assume there exists geodesic arcs γ from a to b and γ' from b to c. Then there exists a geodesic arc from a to c.*
More precisely, the arc γ'' defined by, for $t \in [0, d(a,b)]$, $\gamma''(t) = \gamma(t)$ and for $t \in [d(a,b), d(a,c)]$, $\gamma''(t) = \gamma'(t - d(a,b))$ is a geodesic arc from a to c.

By using Remark 2.2.6 and an immediate induction on the length of the geodesic path considered, we obtain the following corollary :

Corollary 2.2.8. *Let (X,d) be a metric space and $(x_0, ..., x_n)$ be a geodesic path from a to b in X. Assume that, for $i = 0, ..., n-1$, there exists a geodesic arc from x_i to x_{i+1}. Then there exists a geodesic arc from a to b.*

If γ is a continuous arc from a real interval I to X, we denote $\boldsymbol{\gamma} = Im(\gamma)$.

Remark 2.2.9. *Assume (X,d) is a geodesic metric space and denote for $a,b \in X$, $\Gamma_{a\to b}$ the set of geodesic arcs from a to b. We have, for all $a,b \in X$,*

$$[a,b] = \bigcup_{\gamma \in \Gamma_{a\to b}} \boldsymbol{\gamma}.$$

Definition 2.2.10. *Let $a,b,c \in X$ and $\gamma, \gamma', \gamma''$ be geodesic arcs from, respectively, a to b, b to c and c to a. The set*

$$\Delta(a,b,c) = \boldsymbol{\gamma} \cup \boldsymbol{\gamma'} \cup \boldsymbol{\gamma''}$$

is called a geodesic triangle *of the triple a,b,c.*

Definition 2.2.11. *Let (X,d) be a metric space and $a,b,c \in X$. The set*

$$\Delta_f(a,b,c) = [a,b] \cup [b,c] \cup [c,a]$$

is called the full triangle *of a,b,c.*

2.3 Definitions and first properties

Let (X,d) be a metric space.

Definition 2.3.1 (δ-median points)**.** *Let $a, b, c \in X$ and $\delta \geqslant 0$. We denote :*

$$M_\delta(a,b,c) = [a,b]_\delta \cap [b,c]_\delta \cap [c,a]_\delta.$$

The elements of the set $M_\delta(a,b,c)$ are called δ-median points *of the triple a, b, c.*

Definition 2.3.2. *Let $\alpha \geqslant 0$, $a, b, c \in X$ and $\Delta_f(a,b,c)$ be the full triangle of a, b, c.*

We say that $\Delta_f(a,b,c)$ is α-thin *if there exists $x \in [a,b]$, $y \in [b,c]$ and $z \in [c,a]$ pairwise α-close.*
Such elements x, y, z are called α-inner points *of $\Delta_f(a,b,c)$.*

Definition 2.3.3. *Let (X,d) be a metric space.*

- *We say that (X,d) is a L_δ* space *if, for all $a, b, c \in X$, $M_\delta(a,b,c) \neq \emptyset$.*
- *We say that (X,d) is a L'_δ* space *if, for all $a, b, c \in X$, the full triangle $\Delta_f(a,b,c)$ is $\frac{\delta}{2}$-thin.*

Remark 2.3.4. *Let $\delta \geq 0$ and $a, b, c \in X$. Notice that $\frac{\delta}{2}$-inner points of the full triangle $\Delta_f(a,b,c)$ are δ-median points of a, b, c.*
In fact, let $x \in [a,b]$, $y \in [b,c]$ and $z \in [c,a]$ be $\frac{\delta}{2}$-inner points of $\Delta_f(a,b,c)$. Then, we have :

$$d(b,x) + d(x,c) \leqslant d(b,y) + d(y,c) + 2d(x,y) \leqslant d(b,c) + \delta,$$

and, similarly

$$d(c,x) + d(x,a) \leqslant d(c,a) + \delta.$$

Hence $x \in [a,b] \cap [b,c]_\delta \cap [c,a]_\delta$ and hence $x \in M_\delta(a,b,c)$.
It follows that if (X,d) is a L'_δ space then it is a L_δ space.

Proposition 2.3.5. *Let (X,d) be a L'_δ space and $\alpha \geq 0$. For every $a, b \in X$, we have :*

$$[a,b]_\alpha \subset V_{\frac{\alpha}{2}+\frac{3\delta}{4}}([a,b]).$$

Proof. Let $x \in [a,b]_\alpha$. Since (X,d) is L'_δ, the full triangle $\Delta_f(a,b,x)$ is $\frac{\delta}{2}$-thin. Let $a_0 \in [b,x]$, $b_0 \in [x,a]$ and $x_0 \in [a,b]$ be $\frac{\delta}{2}$-inner points of $\Delta_f(a,b,x)$. We have :

$$\begin{array}{lrcl} 1) & d(x,x_0)+d(a_0,b) & \leq & d(x,b)+d(a_0,x_0), \\ 2) & d(x,x_0)+d(a_0,a) & \leq & d(x,a)+d(b_0,x_0)+d(a_0,b_0), \\ 1)+2) & 2d(x,x_0)+d(a,b) & \leq & d(a,b)+\alpha+\frac{3\delta}{2}. \end{array}$$

Since x_0 belongs to $[a,b]$, it follows that $d(x,[a,b]) \leq \frac{\alpha}{2}+\frac{3\delta}{4}$. □

Definition 2.3.6. *Let $\delta \geqslant 0$. We say that a metric space (X,d) is a δ-*median *space if :*

(δ-Med1) *(X,d) is a L'_δ space ;*

(δ-Med2) *there exists $C = C(\delta) \geqslant 0$ such that, for all $a,b,c \in X$,*

$$diam(M_\delta(a,b,c)) \leqslant C\delta.$$

We say that a metric space X is a quasi-median *space if there exists $\delta \geqslant 0$ such that X is δ-median.*

Remark 2.3.7. *A 0-median space is a median metric space. In fact, the L'_0 property, implies that $[a,b]\cap[b,c]\cap[c,a]$ is non empty i.e. there exists a median point for each triple a,b,c, and the condition **(δ-Med2)** gives the unicity of the median point for each triple.*

Remark 2.3.8.

— *If (X,d) is a L_δ space for some $\delta \geq 0$, then it is $L_{\delta'}$ for all $\delta' \geq \delta$ since we have, for all $a,b,c \in X$,*

$$M_\delta(a,b,c) \subset M_{\delta'}(a,b,c).$$

— *If (X,d) is a L'_δ space for some $\delta \geq 0$, then it is $L'_{\delta'}$ for all $\delta' \geq \delta$. In fact, if the full triangle of $a,b,c \in X$ is $\frac{\delta}{2}$-thin then, clearly, it is $\frac{\delta'}{2}$-thin for all $\delta' \geq \delta$.*

By the previous remark, the condition **(δ-Med1)** is satisfied for every $\delta' \geq \delta$; the next result states that the set of δ'-median points of every triple a,b,c stays uniformly close to the set of δ-median points of a,b,c. We will give a full proof of Theorem 7 in Section 2.5.

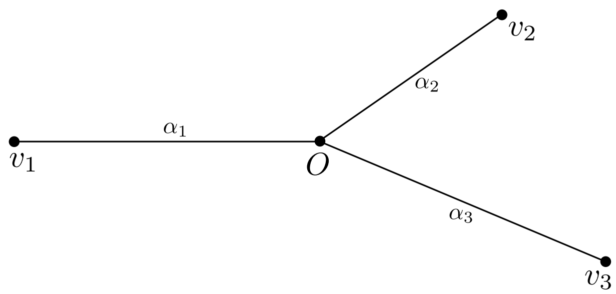
Theorem 7.
Let (X,d) be a δ-median space. Then (X,d) is δ'-median for all $\delta' \geq \delta$.

2.4 Examples of δ-median spaces

2.4.1 Gromov hyperbolic spaces

For our purpose, we consider one of the several equivalent definition of hyperbolicity for metric space that Gromov discussed in [Gro87]. We give here a reformulation of the definition based on tripods given in [BH99] (see Chapter III.H Definition 1.16).

Definition 2.4.1 (Tripod)**.** *Let $\alpha_1, \alpha_2, \alpha_3$ be positive reals. The metric tree $T(\alpha_1, \alpha_2, \alpha_3)$ with tree vertices v_1, v_2, v_3 of valence one, one vertex O of valence 3 and edges of length $d_T(v_1, O) = \alpha_1$, $d_T(v_2, O) = \alpha_2$ and $d_T(v_3, O) = \alpha_3$ is called the* tripod associated with $\alpha_1, \alpha_2, \alpha_3$*. The definition is extended to the cases where at least one of α_i's is zero in the obvious way.*



Definition 2.4.2. *Let (X, d) be a geodesic metric space.*

— *Let $a \in X$. The Gromov product of $b, c \in X$ with respect to a is the quantity :*

$$(b \cdot c)_a = \frac{1}{2}(d(b, a) + d(c, a) - d(b, c)).$$

— *Let $\Delta = \Delta(a, b, c)$ be a geodesic triangle in X, set $\alpha_1 = (b \cdot c)_a$,$\alpha_2 = (c \cdot a)_b$ and $\alpha_3 = (a \cdot b)_c$ and let $T_\Delta = T(\alpha_1, \alpha_2, \alpha_3)$ be the tripod associated with $\alpha_1, \alpha_2, \alpha_3$. We denote by f_Δ the unique map :*

$$f_\Delta : \Delta \to T_\Delta,$$

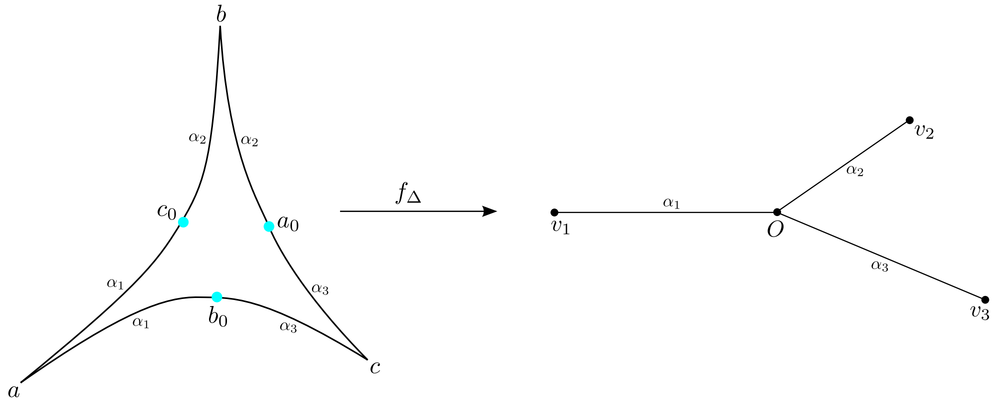
such that $f_\Delta(a) = v_1$,$f_\Delta(b) = v_2$, $f_\Delta(c) = v_3$ and the restriction of f_Δ to each side is an isometry.

The points of $f_\Delta^{-1}(O)$ are called the internal points *of Δ.*

Remark 2.4.3. *Let $\Delta(a, b, c) = \boldsymbol{\gamma_{a,b}} \cup \boldsymbol{\gamma_{b,c}} \cup \boldsymbol{\gamma_{c,a}}$ be a geodesic triangle in X. The unique elements a_0, b_0, c_0 such that :*

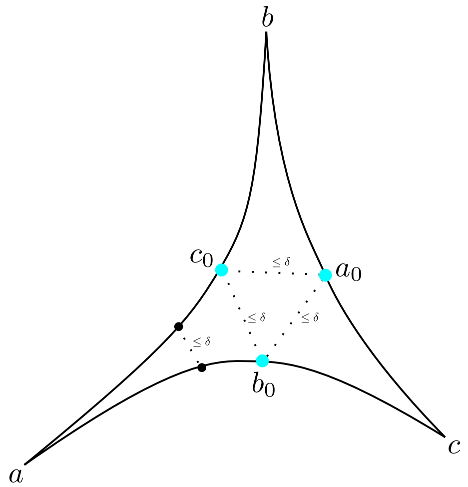
$$a_0 \in \boldsymbol{\gamma_{b,c}} \text{ and } d(b, a_0) = (c \cdot a)_b$$
$$b_0 \in \boldsymbol{\gamma_{c,a}} \text{ and } d(b_0, c) = (a \cdot b)_c$$
$$c_0 \in \boldsymbol{\gamma_{a,b}} \text{ and } d(c_0, a) = (b \cdot c)_a$$

are exactly the internal points *of* $\Delta(a, b, c)$.



The map f_Δ from $\Delta(a, b, c)$ to T_Δ.

Definition 2.4.4. *We say that* (X, d) *is* Gromov hyperbolic *if there exists* $\delta \geq 0$ *such that for every geodesic triangle* $\Delta = \Delta(a, b, c)$ *and for all* $t \in T_\Delta$, $x, y \in f_\Delta^{-1}(t)$ *implies* $d(x, y) \leq \delta$.
For more concision, in this case, we say that (X, d) *is* δ-hyperbolic.



A δ-thin geodesic triangle.

Example 2.4.5. - *Trees, and more generally* $\mathbb{R}$*-trees are* 0*-hyperbolic ;*
- for $\kappa < 0$, *CAT(*κ*) spaces are* δ*-hyperbolic for* d *depending only on* κ.

Proposition 2.4.6. *Let (X, d) be a geodesic metric space. If (X, d) is Gromov hyperbolic then it is a quasi-median space. More precisely, if δ is the hyperbolicity constant of X, then X is 2δ-median with **(2δ-Med2)** constant $C \leq 17$.*

Proof. Assume that X is δ-hyperbolic. Let $a, b, c \in X$, $\Delta(a, b, c)$ be a geodesic triangle in X and a_0, b_0, c_0 be the internal points of $\Delta(a, b, c)$. Since a_0 belongs to a geodesic arc from b to c, then a_0 belongs to the geodesic interval $[b, c]$ and similarly, $b_0 \in [c, a]$ and $c_0 \in [a, b]$. As a_0, b_0, c_0 are pairwise δ-close, it follows that a_0, b_0, c_0 are δ-inner points of the full triangle $\Delta_f(a, b, c)$; and hence, $\Delta_f(a, b, c)$ is δ-thin. As a consequence, X is $L'_{2\delta}$.

Let us show that **(2δ-Med2)** is satisfied. Let $a, b, c \in X$ and $m \in M_{2\delta}(a, b, c)$. Since X is $L'_{2\delta}$, by Proposition 2.3.5, we have :

$$m \in V_{\frac{5\delta}{2}}([a, b]) \cap V_{\frac{5\delta}{2}}([b, c]) \cap V_{\frac{5\delta}{2}}([c, a]).$$

Thus, there exists $m_1 \in [a, b]$, $m_2 \in [b, c]$, $m_3 \in [c, a]$ such that $d(m, m_i) \leq \frac{5\delta}{2}$. As (a, m_1, b), (b, m_2, c), (c, m_3, a) are geodesic path, by Lemma 2.2.7, there exists a geodesic triangle $\Delta(a, b, c) = \boldsymbol{\gamma_1} \cup \boldsymbol{\gamma_2} \cup \boldsymbol{\gamma_3}$ such that $m_i \in \boldsymbol{\gamma_i}$ where $\gamma_1, \gamma_2, \gamma_3$ are geodesic path from a to b, from b to c and from c to a respectively. Let $a_0 \in \boldsymbol{\gamma_2}$, $b_0 \in \boldsymbol{\gamma_3}$, $c_0 \in \boldsymbol{\gamma_1}$ be the internal points of $\Delta(a, b, c)$. Notice that there is at most two points among m_1, m_2, m_3 such that their images by f_Δ belong to the same edge of T_Δ. Hence, we can assume, up to permute a, b and c that $f_\Delta(m_1)$ belong to the edge (v_1, O) and $f_\Delta(m_2)$ belong to the edge (v_2, O). Let m'_2 be the unique point in $f_\Delta^{-1}(m_2) \cap \boldsymbol{\gamma_1}$. Since the restriction of f_Δ to $\boldsymbol{\gamma_1}$ is an isometry, we have :

$$d(m_1, c_0) + d(c_0, m'_2) = d(m_1, m'_2).$$

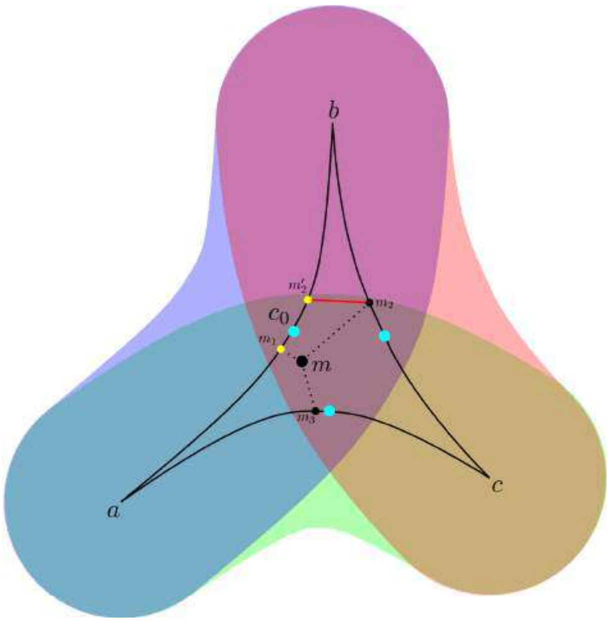
By δ-hyperbolicity, $d(m_2, m'_2) \leq \delta$ and then :

$$d(m_1, c_0) \leq d(m_1, m'_2) \leq d(m_1, m_2) + d(m_2, m'_2) \leq 2 \times \frac{5\delta}{2} + \delta = 6\delta,$$

and hence, $d(m, c_0) \leq d(m, m_1) + d(m_1, c_0) \leq \frac{17\delta}{2}$. It follows that

$$diam(M_{2\delta}(a, b, c)) \leq 17\delta.$$

□



2.4.2 A construction of δ-median space

Let $\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ be the Poincaré half-plane model i.e. the geodesic metric space $(\mathbb{H}, d_{\mathbb{H}})$ where, for $z = x + iy, z' = x' + iy' \in \mathbb{H}$,

$$d_{\mathbb{H}}(z, z') = \operatorname{arcosh}\left(1 + \frac{(x - x')^2 + (y - y')^2}{2yy'}\right);$$

The space $(\mathbb{H}, d_{\mathbb{H}})$ is a uniquely geodesic δ-hyperbolic space for $\delta = \ln(\frac{3+\sqrt{5}}{2})$. The geodesic arcs in $\mathbb{H}$ are exactly arcs of circles having a diameter contained in the real axis or segments of lines perpendicular to the real axis.

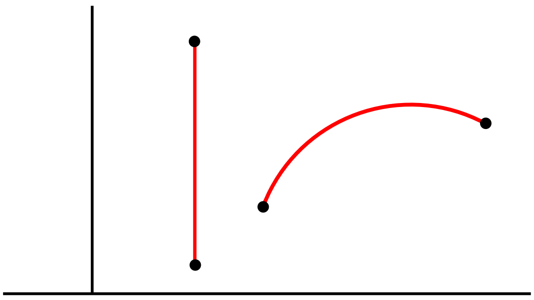


Figure 1.
Examples of geodesic arcs in $(\mathbb{H}, d_{\mathbb{H}})$

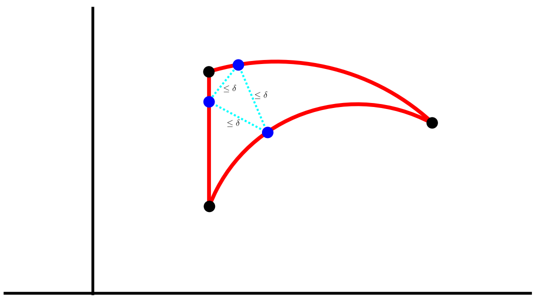


Figure 2.
A slim triangle in $\mathbb{H}$.

Let $\mathbb{R}^2 = \{(x,y)|x,y \in \mathbb{R}\}$ be the real plane endowed with the metric induced by the norm 1 on $\mathbb{R}^2$:

$$d_{\|.\|_1}((x,y),(x',y')) = |x-x'| + |y-y'|.$$

The space $(\mathbb{R}^2, d_{\|.\|_1})$ is a median space and for $\delta \geq 0$, the **(δ-Med2)** constant C of $(\mathbb{R}^2, d_{\|.\|_1})$ (seen as a δ-median space) is $C = 3$ (see Remark 2.5.8). Notice that, with this metric, $\mathbb{R}^2$ is not Gromov hyperbolic : the distance between some pairs of internal points of the geodesic triangle in Figure 4. below, goes to infinity when $n \to +\infty$.

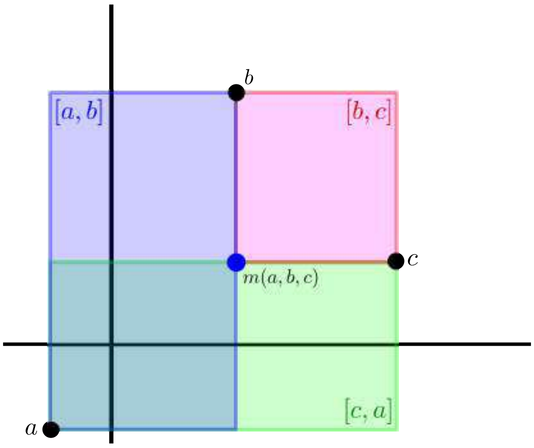


Figure 3.
Geodesic intervals and median point

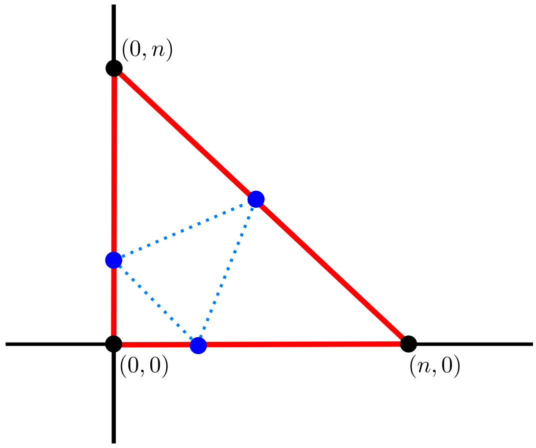


Figure 4.
Sequence of non slim triangles.

1. A non-hyperbolic non-median quasi-median space

In this part, from the hyperbolic half-plane and an isometric copy of $\mathbb{R}^2$ endowed with the ℓ^1 metric, we give an explicit construction of a non-hyperbolic and non-median quasi-median space.

Fix a non-negative real number L and consider a geodesic ars γ in $\mathbb{H}$ of length L, i.e.

$$\gamma : [0, L] \to \mathbb{H} \text{ is an isometry,}$$

To this arc γ, we associate an isometric copy $(\mathcal{P}, d_1)$ of $(\mathbb{R}^2, d_{\|.\|_1})$ and we build a metric space (X, d) as follows :

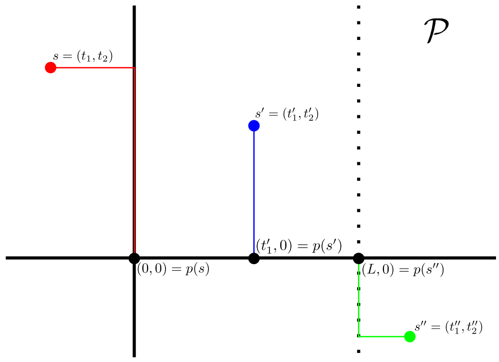
- Define the identification $\sim$ by : we glue isometrically the segment $[(0,0),(L,0)]$ of

$\mathcal{P} = \{(t, t') \mid t, t' \in \mathbb{R}\}$ on $\boldsymbol{\gamma}$ by identifying, for $t \in [0, L]$,

$$(t, 0) \sim \gamma(t).$$

- Set $X = (\mathbb{H} \sqcup \mathcal{P})/\sim$ and let $\pi : \mathbb{H} \sqcup \mathcal{P} \to X$ be the canonical surjection.
- Consider the projection p on the segment $[(0,0), (L,0)]$ of $\mathcal{P}$ such that, for $(t, t') \in \mathcal{P}$, $p(t, t') = (f(t), 0) \in \mathcal{P}$ where $f : \mathbb{R} \to [0, L]$ is defined by :

$$f(t) = \begin{cases} t & \text{if } t \in [0, L], \\ 0 & \text{if } t < 0, \\ L & \text{if } t > L; \end{cases}$$



and define a map $d : X \times X \to \mathbb{R}_+$ by, for $a, b \in X$:
— if $a = \pi(z)$, $b = \pi(z')$ with $z, z' \in \mathbb{H}$,

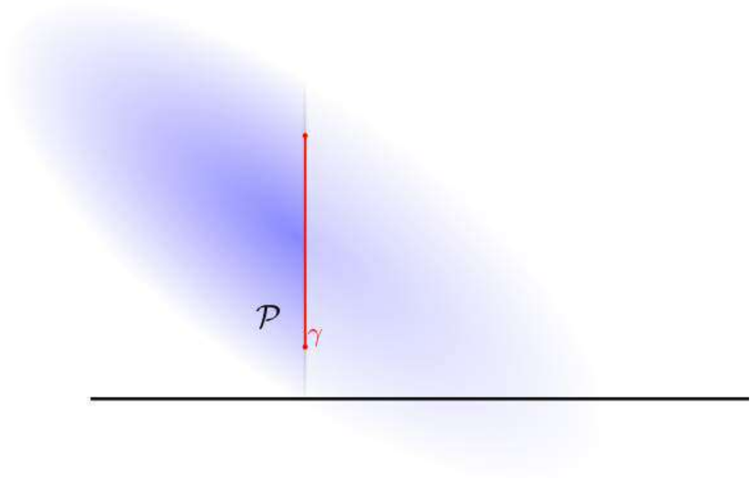
$$d(a, b) = d_{\mathbb{H}}(z, z');$$

— if $a = \pi(s)$, $b = \pi(s')$ with $s, s' \in \mathcal{P}$,

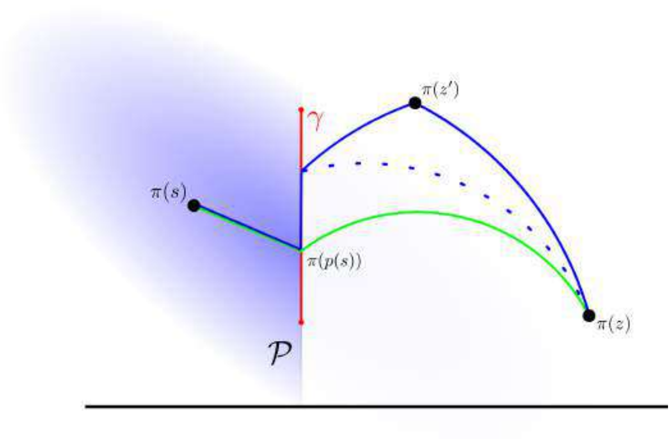
$$d(a, b) = d_1(s, s');$$

— if $a = \pi(z)$, $b = \pi(s)$ (or vice versa) with $z \in \mathbb{H}$ and $s = (t, t') \in \mathcal{P}$,

$$d(a, b) = d_{\mathbb{H}}(z, \gamma(t)) + d_1(p(s), s);$$



$X = (\mathbb{H} \sqcup \mathcal{P})/\sim$ for a given geodesic arc γ.



A geodesic triangle in X for the metric d.

Proposition 2.4.7. *The couple (X, d) defined above is a geodesic metric space.*

Proof. First, we claim that d define a metric on X. In fact, notice that, for $a = \pi(z)$, $b = \pi(s)$ with $z \in \mathbb{H}$ and $s = (t, t') \in \mathcal{P}$,

$$d(a, b) = d_{\mathbb{H}}(z, \gamma(t)) + d_1(p(s), s) = \inf_{T \in [0,L]} (d(z, \gamma(T)) + d_1((T, 0), s))$$

by triangular inequality of $d_{\mathbb{H}}$ (see the figure above). Moreover, when restricted to each $\pi(\mathcal{P})$ or to $\pi(\mathbb{H})$, d "extends" the metrics of each $\mathcal{P}$ or $\mathbb{H}$. Hence, the triangular inequality for d is satisfied on X and it is clear that the separation axiom is satisfied too from the definition.

Let us show that (X, d) is a geodesic.

1) Let $a = \pi(z)$, $b = \pi(z')$ with $z, z' \in \mathbb{H}$. Then, there exists a geodesic arc α in $\mathbb{H}$ from z to z' and the arc $\tilde{\alpha}$ defined by $\tilde{\alpha} = \pi \circ \alpha$ satisfies, for $0 \leq t, t' \leq d_{\mathbb{H}}(z, z') = d(a, b)$:

$$d(\tilde{\alpha}(t), \tilde{\alpha}(t')) = d_{\mathbb{H}}(\alpha(t), \alpha(t')) = |t - t'|,$$

is a geodesic arc from a to b in X.

2) Let $a = \pi(s)$, $b = \pi(s')$ with $s, s' \in \mathcal{P}$. Then, there exists a geodesic arc β in $\mathcal{P}$ from s to s' and the arc $\tilde{\beta}$ defined by $\tilde{\beta} = \pi \circ \beta$ satisfies, for $0 \leq t, t' \leq d_1(s, s') = d(a, b)$:

$$d(\tilde{\beta}(t), \tilde{\beta}(t')) = d_1(\beta(t), \beta(t')) = |t - t'|,$$

is a geodesic arc from a to b in X.

3) Let $a = \pi(z)$, $b = \pi(s)$ (or vice versa) with $z \in \mathbb{H}$ and $s = (t, t') \in \mathcal{P}$. Set $c = \pi(p(s)) (= \pi(\gamma(t)))$ and notice that :

$$d(a, c) + d(c, b) = d_{\mathbb{H}}(z, \gamma(t)) + d_1(p(s), s) = d(a, b),$$

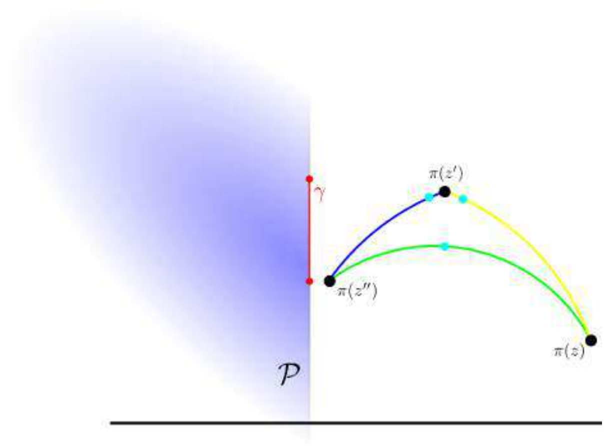
and hence, c belong to the geodesic interval $[a, b]$. By 1) and 2), there exists geodesic arcs in X from a to c and from c to b and then, by Lemma 2.2.8, there exists a geodesic arc from a to b in X.

It follows that (X, d) is a geodesic metric space.

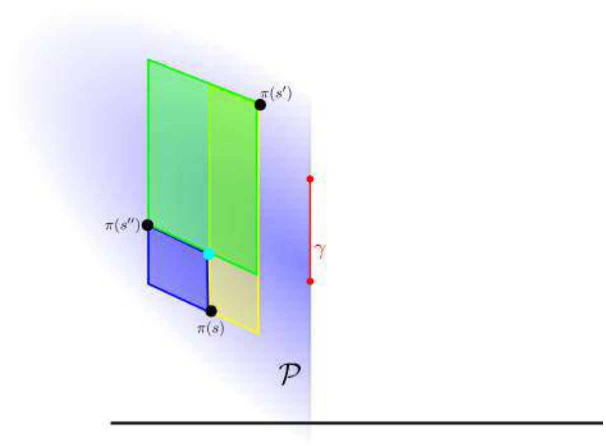
□

Notice that $\pi_{|\mathbb{H}}$ is an isometry from $\mathbb{H}$ to X and $\pi_{|\mathcal{P}}$ is an isometry from $\mathcal{P}$ to X.

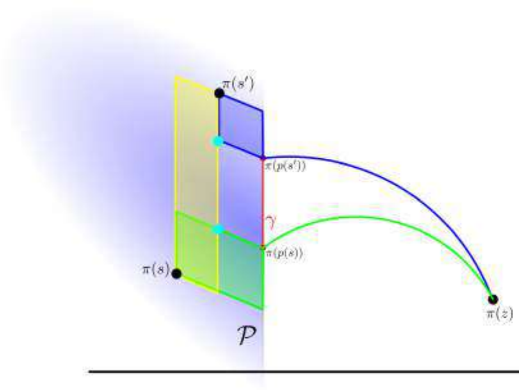
Let $\delta = \max(2L, 2\ln(\frac{3+\sqrt{5}}{2}))$. There are four generic full triangles in X and notice that each of those full triangles is $\frac{\delta}{2}$-thin : in each picture below, the points colored in sky blue are some $\frac{\delta}{2}$-inner points of the full triangle of the three elements of X considered.



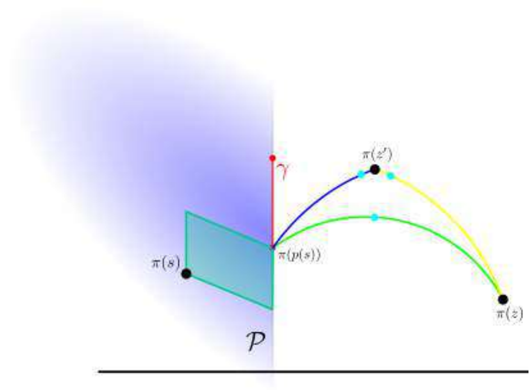
Triangle 1



Triangle 2



Triangle 3



Triangle 4

Proposition 2.4.8. *The space* (X, d) *is* δ*-median for* $\delta = \max(2L, 2ln(\frac{3+\sqrt{5}}{2}))$.

Proof. Let $\delta = \max(2L, 2\ln(\frac{3+\sqrt{5}}{2}))$. If $\Delta_f(a,b,c)$ is a full triangle in X, there is different three cases (up to permute a, b and c) for the location of the inner points of $\Delta_f(a,b,c)$:

First case : $a, b, c \in \pi(\mathbb{H})$ or there exists $i \in \{1, ..., n\}$ such that $a, b, c \in \pi(\mathcal{P})$ (Triangles 1 and 2).
Then the full triangle $\Delta_f(a,b,c)$ of a, b, c is isometric to a full triangle in $\mathbb{H}$ or in $\mathcal{P}$ which is $\frac{\delta}{2}$-thin as $\mathbb{H}$ and $\mathcal{P}$ are L'_δ. Hence, $\Delta_f(a,b,c)$ is δ-thin.

Second case : a, b, c do not pairwise belong to $\pi(\mathcal{P})$, (Triangles 4).
Then there exists $z_a, z_b, z_c \in \mathbb{H}$ such that :

$$\Delta_f(\pi(z_a), \pi(z_b), \pi(z_c)) \subset \Delta_f(a,b,c).$$

The full triangle $\Delta_f(\pi(z_a), \pi(z_b), \pi(z_c))$ is isometric to $\Delta_f(z_a, z_b, z_c)$ which is $\frac{\delta}{2}$-thin triangle in $\mathbb{H}$. Hence, for any triple $z'_a \in [z_b, z_c], z'_b \in [z_c, z_a], z'_b \in [z_a, z_b]$ of $\frac{\delta}{2}$-inner points of $\Delta_f(z_a, z_b, z_c)$, $\pi(z'_a) \in [b,c]$, $\pi(z'_b) \in [c,a]$,$\pi(z'_c) \in [a,b]$ are $\frac{\delta}{2}$-close. It follows that $\Delta_f(a,b,c)$ is $\frac{\delta}{2}$-thin.

Third case : $a, b \in \pi(\mathcal{P})$ and $c \notin \pi(\mathcal{P})$, (Triangles 3).
Then, there exists $z_c \in \mathbb{H}$ such that :

$$\Delta_f(a, b, \pi(z_c)) \subset \Delta_f(a,b,c).$$

Let $s_a = (t_a, t'_a), s_b = (t_b, t'_b) \in \mathcal{P}$ such that $a = \pi(s_a)$, $b = \pi(s_b)$ and consider $p(s_a) = (p(t_a), 0)$, $p(s_b) = (p(t_b), 0)$ in $\mathcal{P}$.

If the signs t'_a and t'_b are opposite, then $[\pi(p(s_a)), \pi(p(s_b))] \subset \Delta_f(a, b, \pi(z_c))$ and hence, $\pi(p_i(s_a)) \in [a,b], \pi(p(s_a)) \in [\pi(z_c), a] \subset [c,a]$, and $\pi(p(s_b)) \in [b, \pi(z_c)] \subset [b,c]$ are L-close, and hence $\Delta_f(a,b,c)$ is δ-thin.

If t'_a, t'_b have the same sign, we can assume, without loss of generality that they are non-negative and we set : $t' = \min(t'_a, t'_b)$ and

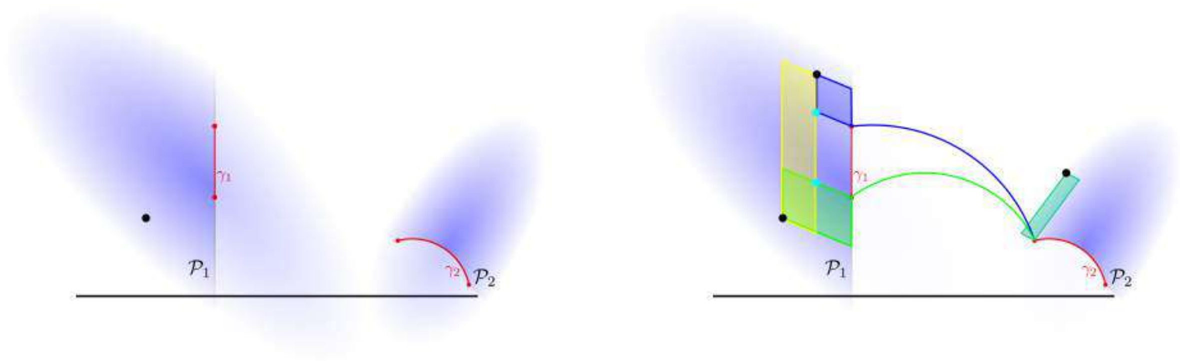
$$s'_a = (f(t_a), t') \text{ and } s'_b = (f(t_b), t').$$

We have : $d(\pi(s'_a), \pi(s'_a)) = |f(t_a) - f(t_b)| \leq L \leq \frac{\delta}{2}$ and $\pi(s'_a) \in [c,a]$, $\pi(s'_b) \in [b,c]$ and $\pi(s'_a) \in [a,b]$. Hence, $\Delta_f(a,b,c)$ is $\frac{\delta}{2}$-thin.

As a conclusion, (X,d) is L'_δ.

$\square$

Remark 2.4.9. *This construction can be easily generalized by considering a finite collection of pairwise non intersecting geodesic arcs and, on each arc, we glue in the same way a isometric copy of* $(\mathbb{R}^2, d_{\|.\|_1})$*. By similar arguments, the resulting metric space is quasi-median.*



2. A L_δ space which is not L'_δ

Here, by a similar construction using the Poincare half-plan model and an isometric copy of $\mathbb{R}^2$ with the ℓ^1-metric, we build a L_δ space which has sequences of full triangles that do not satisfy the α-thinness condition for any $\alpha \geq 0$.

Consider a lines γ in $\mathbb{H}$ i.e.

$$\gamma : \mathbb{R} \to \mathbb{H} \text{ is an isometry .}$$

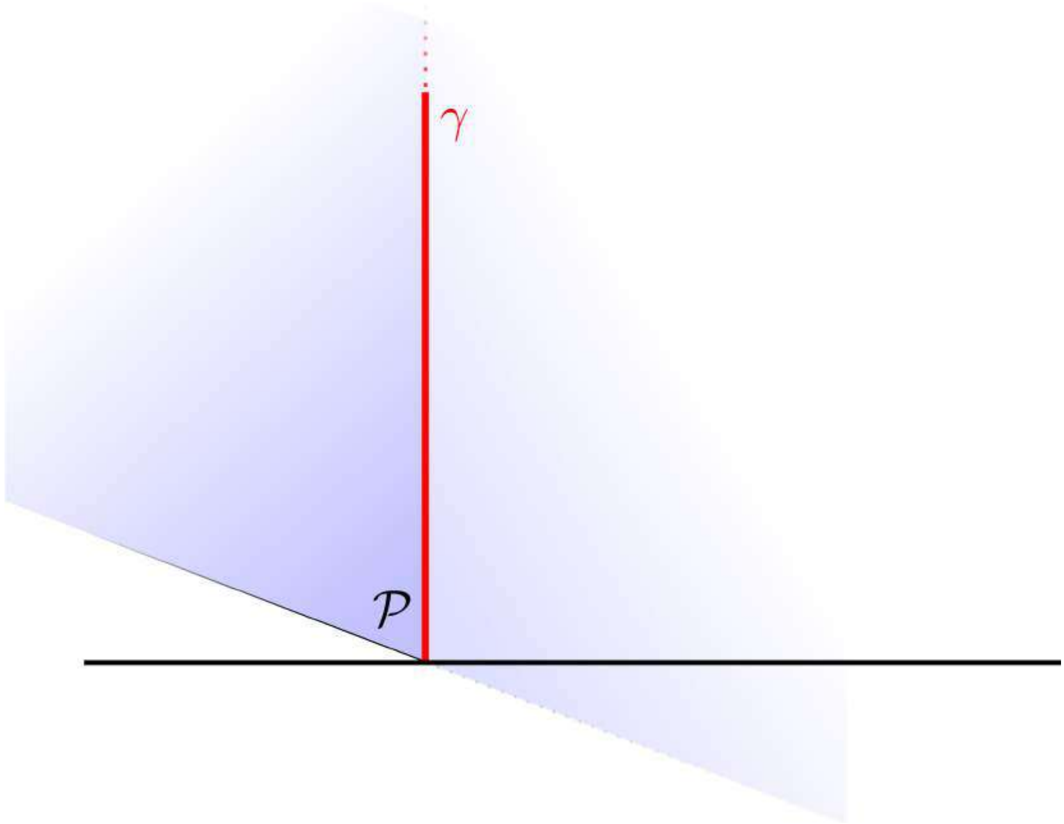
We build a geodesic metric space (X,d) as previously : we consider an isometric copy $(\mathcal{P}, d_1)$ of $(\mathbb{R}^2, d_{\|\|\|_1})$ and we glue the real axis $\{(t,0) \mid t \in \mathbb{R}\}$ and $\boldsymbol{\gamma}$ by the following identification : for $t \in \mathbb{R}$,

$$(t,0) \sim \gamma(t).$$

The metric d on X is defined as previously, but we replace the projection p by the canonical projection of $\mathcal{P}$ on its absciss axis :

$$p(t, t') = (t, 0).$$

Using similar arguments as in proof of Proposition 2.4.7, we can see that the space (X, d) is a geodesic metric space.



$X = (\mathbb{H} \sqcup \mathcal{P})/\sim$ for a given geodesic line γ.

Proposition 2.4.10. *The space* (X, d) *is* L_δ *for* $\delta = 2ln(\frac{3+\sqrt{5}}{2})$.

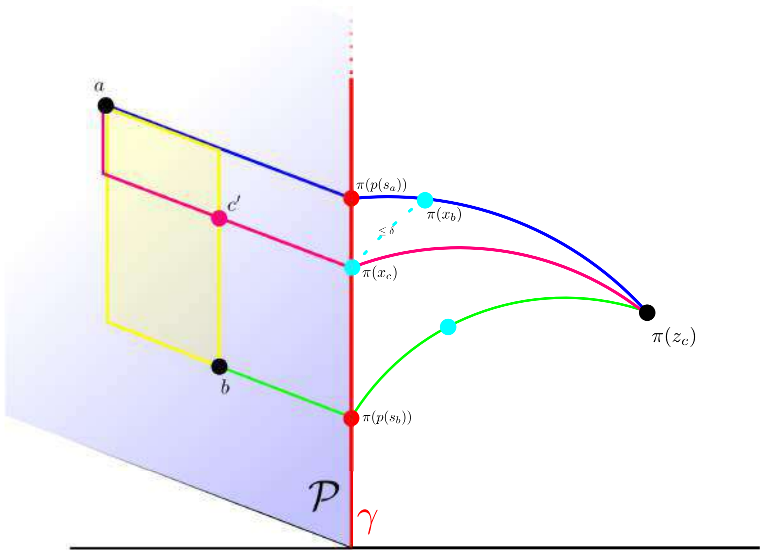
Proof. Let $a, b, c \in X$. If either $a, b, c \in \pi(\mathcal{P})$ or if a, b, c do not pairwise belong to $\pi(\mathcal{P})$, then, as in proof of Proposition 2.4.8, there exists a δ-thin subtriangle of $\Delta_f(a, b, c)$ and hence, $\Delta_f(a, b, c)$ admits δ-inner points. Thus $M_\delta(a, b, c)$ is not empty.

It remains (up to permute a, b, c) the case where $a, b \in \pi(\mathcal{P})$ and $c \notin \pi(\mathcal{P})$. Then there exists $z_c \in \mathbb{H}$ such that $M_\delta(a, b, \pi(z_c)) \subset M_\delta(a, b, c)$. Let $s_a = (t_a, t'_a), s_b = (t_b, t'_b) \in \mathcal{P}$ such that $a = \pi(s_a)$, $b = \pi(s_b)$ and consider $p(s_a) = (t_a, 0)$, $p(s_b) = (t_b, 0)$ in $\mathcal{P}$. By the identification $\sim$, we have $\pi(p(s_a)) = \pi(\gamma(t_a))$, $\pi(p(s_b)) = \pi(\gamma(t_b))$ and we consider the geodesic triangle $\Delta = \Delta(\gamma(t_a), \gamma(t_b), z_c)$ in $\mathbb{H}$. As $\mathbb{H}$ is $\frac{\delta}{2}$-hyperbolic, Δ admits $\frac{\delta}{2}$-close internal points $x_a \in [\gamma(t_b), z_c] \subset [b, c]$, $x_b \in [\gamma(t_a), z_c] \subset [a, c]$ and $x_c \in [\gamma(t_a), \gamma(t_b)] \subset [a, b]$.

If the signs of t'_a and t'_b are opposite, then we have $\pi(x_c) \in [\pi(\gamma(t_a)), \pi(\gamma(t_b))] \subset [a, b]$. And hence,

$$\pi(x_c) \in [a, b] \cap [b, c]_\delta \cap [c, a]_\delta \subset M_\delta(a, b, c).$$

If t'_a, t'_b have the same sign, we can assume without loss of generality, that t'_a, t'_b are both non-negative.



We have $\pi(x_c) \in [\pi(\gamma(t_a)), \pi(\gamma(t_b))]$, then there exists $t_c \in [t_a, t_b]$ such that $\pi(x_c) = \pi((t_c, 0))$. Set $t' = \min(t'_a, t'_b)$ and

$$c' \in \pi(\mathcal{P}) \text{ such that } c' = \pi((t_c, t')).$$

We claim that $c' \in [b, c]_\delta \cap [c, a]_\delta$. In fact, we have :

$$\begin{aligned} d(a,\pi(x_c))+d(\pi(x_c),\pi(z_c)) &\leq d(a,\pi(x_b))+2d(\pi(x_b),\pi(x_c))+d(\pi(x_b),\pi(z_c)),\\ &\leq d(a,\pi(z_c))+\delta. \end{aligned}$$

Hence, $\pi(x_c) \in [a,\pi(z_c)]_\delta \subset [a,c]_\delta$ and, as $c' \in [a,\pi(x_c)]$, we have :

$$c' \in [c,a]_\delta;$$

and similarly, $c' \in [b,c]_\delta$ which proves our claim.
Since $c' \in [a,b]$, it follows that :

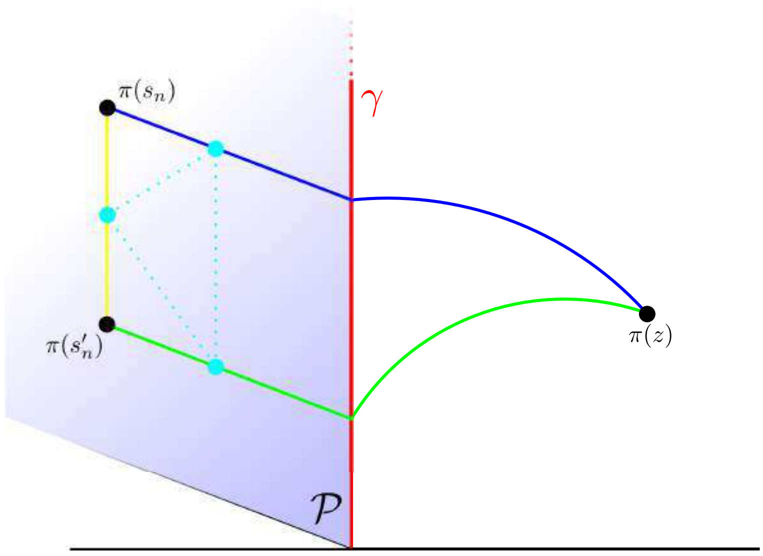
$$c' \in [a,b] \cap [b,c]_\delta \cap [c,a]_\delta \subset M_\delta(a,b,c).$$

Thus, $M_\delta(a,b,c)$ is not empty.

As a conclusion, (X,d) is L_δ. $\square$

Proposition 2.4.11. *For all $\alpha \geq 0$, (X,d) is not L'_α.*

Proof. For $n \in \mathbb{N}$, we set $s_n = (n,n) \in \mathcal{P}$ and $s'_n = (-n,n) \in \mathcal{P}$. Pick an element $z \in \mathbb{H} \setminus \{\boldsymbol{\gamma}\}$ and denote by Δ_n the full triangle of $\pi(s_n),\pi(s'_n),\pi(z)$.



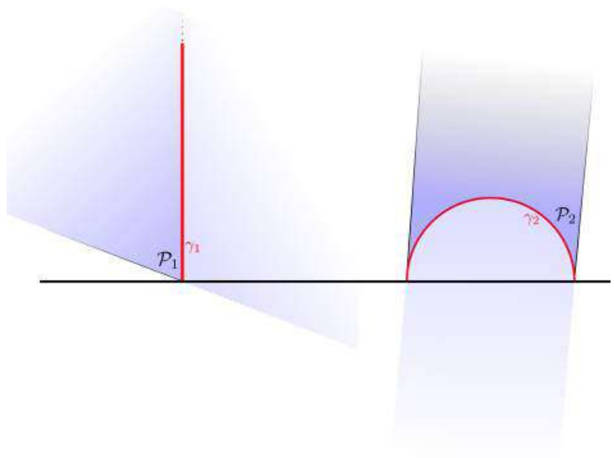
The full triangle Δ_n.

Let $\alpha \geq 0$ and $n \in \mathbb{N}$ such that $n > \alpha$. Consider the full triangle Δ_n of $\pi(s_n),\pi(s'_n),\pi(z)$. The elements of $[\pi(s_n),\pi(z)]$ and $[\pi(s'_n),\pi(z)]$ which are α-close to $[\pi(s_n),\pi(s'_n)]$ must lie in $V_\delta([\pi(s_n), \pi(s'_n)])$ and we have, for $a \in V_\delta([\pi(s_n), \pi(s'_n)]) \cap [\pi(s_n), \pi(z)]$ and $b \in V_\delta([\pi(s_n), \pi(s'_n)]) \cap [\pi(s'_n), \pi(z)]$:

$$d(a, b) = n > \alpha.$$

And hence, Δ_n is not α-thin. It follows that (X, d) is not L'_α. $\square$

Remark 2.4.12. *As in Remark 2.4.9, this construction can be generalized for a finite collection of pairwise non intersecting geodesic lines.*



The space $X = (\mathbb{H} \sqcup \bigsqcup_{i=1,2} \mathcal{P}_i)/\sim$ for two given geodesic lines.

2.5 Proof of Theorem 7

2.5.1 Gates

Let (X, d) be a metric space.

Definition 2.5.1. *Let $\alpha \geq 0$, $x \in X$ and $Y \subset X$.*
We say that $p \in X$ is α-between x and Y *if p is α-between x and y for all y in Y, i.e. for all $y \in Y$:*

$$p \in [x, y]_\alpha.$$

An element $p \in Y$ which is α-between x and Y is called a α-gate between x and Y. *We denote by $\mathcal{G}_\alpha(x, Y)$ the set of all α-gates between x and Y, i.e.*

$$\mathcal{G}_\delta(x, Y) = \{p \in Y \mid \forall y \in Y,\ p \in [x, y]_\alpha\}.$$

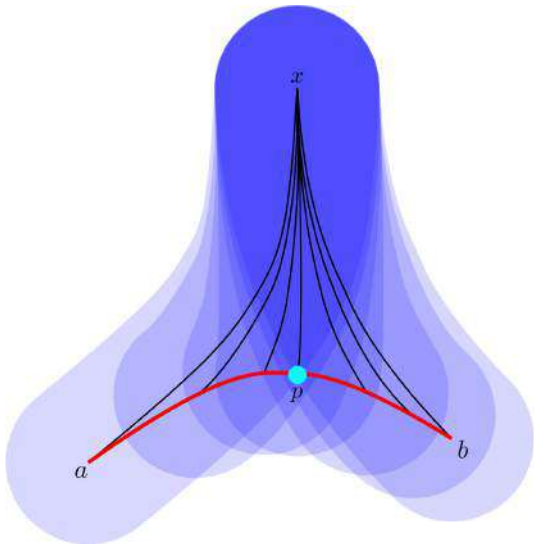


Figure : p is a δ-gate between x and $[a, b]$.

Proposition 2.5.2. *Let $\alpha \geq 0$, $x \in X$ and $Y \subset X$. Then :*

$$diam(\mathcal{G}_\alpha(x, Y)) \leqslant 2\alpha,$$

and if $p \in \mathcal{G}_\alpha(x, Y)$, $d(x, p) \leqslant d(x, Y) + \alpha$.

Proof. Let p be a α-gate between x and Y. For all $y \in Y$, we have $d(x, p) + d(p, y) \leqslant d(x, y) + \alpha$ and then :

$$d(x, p) \leqslant d(x, Y) + \alpha.$$

Moreover, if $p_1, p_2 \in \mathcal{G}_\alpha(x, Y)$, we have :

$$d(x, Y) + d(p_1, p_2) \leqslant d(x, p_1) + d(p_1, p_2) \leqslant d(x, p_2) + \alpha \leqslant d(x, Y) + 2\alpha.$$

It follows that $d(p_1, p_2) \leqslant 2\alpha$. □

Definition 2.5.3. *Let $\alpha \geq 0$. A subset Y of X is said* α-gate convex *if, for all $x \in X$, $\mathcal{G}_\alpha(x, Y)$ is non-empty.*

Lemma 2.5.4. *Let $\alpha \geq 0$, Y be a α-gate convex subset of X and $x, x' \in X$. For all $p \in \mathcal{G}_\alpha(x, Y)$ and all $p' \in \mathcal{G}_\alpha(x', Y)$, we have :*

$$d(p, p') \leq d(x, x') + \alpha.$$

Proof. Since p, p' belong to Y, we have the following inequalities :
 1) $d(x, p) + d(p, p') \leq d(x, p') + \alpha \leq d(x, x') + d(x', p') + \alpha$,
 2) $d(x', p') + d(p', p) \leq d(x', p) + \alpha \leq d(x', x) + d(x, p) + \alpha$.
The result is obtained by computing $\frac{1}{2}$(1)+2)). □

2.5.2 Gates in quasi-median spaces

In this part, we investigate the notion of gate and gate-convexity in the setting of quasi-median spaces and we give a proof of Theorem 7 by using the fact that intervals are quasi-gate-convex in a quasi-median space.
We now assume that (X, d) is a δ-median space with **(δ-Med2)** constant denoted by C.

Proposition 2.5.5. *Let $a, b \in X$. The interval $[a, b]$ is $2(C+1)\delta$-gate convex.*
Moreover precisely, if $x \in X$, every $\frac{\delta}{2}$-inner point of $\Delta_f(x, a, b)$ in $[a, b]$ belongs to $\mathcal{G}_{2(C+1)\delta}(x, [a, b])$.

Proof. Let $x \in X$ and $p \in [a, b]$ be a $\frac{\delta}{2}$-inner point of $\Delta(x, a, b)$. Let us show that $p \in \mathcal{G}_{2(C+1)\delta}(x, [a, b])$.
Let $y \in [a, b]$ and consider $a' \in [x, a]$ a $\frac{\delta}{2}$-inner point of $\Delta_f(x, a, y)$ and $p' \in [a', b]$ a $\frac{\delta}{2}$-inner point of $\Delta_f(x, a', b)$.
First, notice that p' is 2δ-between x and y ; indeed, since $p' \in [x, a']_\delta$ and $a' \in [x, y]_\delta$, we have :

$$d(x, p') + d(p', y) \leq d(x, p') + d(p', a') + d(a', y) \leq d(x, y) + 2\delta.$$

Moreover, p' satisfies :

— $p' \in [x, b]_\delta$, by definition of p' ;

— $p' \in [x, a]_\delta$: since $p' \in [x, a']_\delta$ and $a' \in [x, a]$, we have :

$$d(x, p') + d(p', a) \leq d(x, p') + d(p', a') + d(a', a) \leq d(x, a) + \delta;$$

— $p' \in [a, b]_\delta$: since $p' \in [a', b]$, $a' \in [a, y]_\delta$ and $y \in [a, b]$, we have :

$$d(a, p') + d(p', b) \leq d(a, a') + d(a', p') + d(p', b) \leq d(a, a') + d(a', y) + d(y, b) \leq d(a, b) + \delta.$$

It follows that $p' \in [x,a]_\delta \cap [x,b]_\delta \cap [a,b]_\delta = M_\delta(x,a,b)$ and hence, $d(p,p')$ is less than $C\delta$ by **(δ-Med2)**. Thus, we have :

$$d(x,p) + d(p,y) \leq d(x,p') + d(p',y) + 2d(p',p) \leq d(x,y) + 2C\delta + 2\delta;$$

which means that p is $2(C+1)\delta$-between x and y. □

Proposition 2.5.6. *Let $\delta' \geq \delta$. There exists $K = C(\delta, \delta', C) \geq 0$ such that, for every $a, b, c \in X$ and every $m \in M_{\delta'}(a,b,c)$, there exists $m_0 \in M_\delta(a,b,c)$ such that :*

$$d(m, m_0) \leq K.$$

Proof. Let $a, b, c \in X$ and $m \in M_{\delta'}(a,b,c)$. Consider the following elements in X :
— let $p_1 \in [a,b]$ be a $\frac{\delta}{2}$-inner point of $\Delta_f(m,a,b)$;
— let $p_2 \in [b,c]$ be a $\frac{\delta}{2}$-inner point of $\Delta_f(m,b,c)$;
— let $p_3 \in [c,a]$ be a $\frac{\delta}{2}$-inner point of $\Delta_f(m,c,a)$;
— let $q \in [b,c]$ be a $\frac{\delta}{2}$-inner point of $\Delta_f(p_1,b,c)$ and
— let $m_0 \in [q,a]$, $r \in [c,a]$ be $\frac{\delta}{2}$-inner points of $\Delta_f(q,c,a)$.
We claim that m_0 belong to $M_\delta(a,b,c)$. Indeed, there exists $x \in [q,c]$ such that $d(m_0,x) \leqslant \delta/2$, thus $x \in [b,c]$ as q belongs to $[b,c]$ and hence, $m_0 \in [b,c]_\delta \cap [c,a]_\delta$. Moreover, q belongs to $[a,b]_\delta$ since $q \in [p_1,b]_\delta$ and $p_1 \in [a,b]$. It follows that :

$$d(a,m_0) + d(m_0,b) \leqslant d(a,m_0) + d(m_0,q) + d(q,b) = d(a,q) + d(q,b) \leqslant d(a,b) + \delta,$$

and hence, $m_0 \in [a,b]_\delta \cap [b,c]_\delta \cap [c,a]_\delta$ which proves our claim.

We now show that the distance $d(m,m_0)$ does not depend on a, b, c. First, notice that, by Proposition 2.5.5, $p_1 \in \mathcal{G}_{2(C+1)\delta}(m,[a,b])$, $p_2 \in \mathcal{G}_{2(C+1)\delta}(m,[b,c])$ and $p_1 \in \mathcal{G}_{2(C+1)\delta}(m,[c,a])$, and set :

$$K_0 = K_0(\delta, \delta', C) := \frac{\delta'}{2} + \frac{3\delta}{4} + 2(C+1)\delta.$$

By Lemma 2.3.5, we have $m \in [a,b]_{\delta'} \subset V_{\frac{\delta'}{2}+\frac{3\delta}{4}}([a,b])$ and then, using Proposition 2.5.2, we have the following inequality :

$$d(m,p_1) \leq d(m,[a,b]) + 2(C+1)\delta \leq \frac{\delta'}{2} + \frac{3\delta}{4} + (2C+1)\delta = K_0,$$

and similarly, we have : $d(m,p_2) \leq K_0$ and $d(m,p_3) \leq K_0$.

Since, by Proposition 2.5.5, $p_2 \in \mathcal{G}_{2(C+1)\delta}(m, [b, c])$ and $q \in \mathcal{G}_{2(C+1)\delta}(p_1, [b, c])$, we obtain, by Lemma 2.5.4, that $d(p_2, q) \leq d(m, p_1) + 2(C + 1)\delta = K_0 + 2(C + 1)\delta$ and similarly, we have $d(p_3, r) \leq d(m, q) + 2(C + 1)\delta$. Then, finally :

$$\begin{aligned}
d(m, m_0) &\leq d(m, p_3) + d(p_3, r) + d(r, m_0) \\
&\leq K_0 + d(m, q) + 2(C + 1)\delta + \delta/2 \\
&\leq d(m, p_2) + d(p_2, q) + K_0 + 2(C + 1)\delta + \delta/2 \\
&\leq K_0 + K_0 + 2(C + 1)\delta + K_0 + 2(C + 1)\delta + \delta/2 \\
d(m, m_0) &\leq 3K_0 + (4C + \tfrac{9}{2})\delta.
\end{aligned}$$

It follows that, by setting :

$$K = K(\delta', \delta, C) := 3K_0 + (4C + \frac{9}{2})\delta = \frac{3\delta'}{2} + (10C + \frac{51}{4})\delta,$$

we obtain that any δ'-median point of a, b, c is at distance at most K of a δ-median point of a, b, c. $\square$

Corollary 2.5.7. *Let $\delta' \geq \delta$. There exists $C' = C'(\delta, \delta', C) \geq 0$ such that, for every $a, b, c \in X$,*

$$diam(M_{\delta'}(a, b, c)) \leq C'\delta'.$$

Proof. Let $m, m' \in M_{\delta'}(a, b, c)$. Then, by Proposition 2.5.6, there exists $K \geq 0$ depending only on the constants δ, δ' and C, and there exists $m_0, m_0' \in M_\delta(a, b, c)$ such that $d(m, m_0) \leq K$ and $d(m', m_0') \leq K$. Since (X, d) is δ-median, by **(δ-Med2)**, we have $d(m_0, m_0') \leq C\delta$. It follows that :

$$d(m, m') \leq d(m, m_0) + d(m_0, m_0') + d(m_0', m') \leq 2K + C\delta.$$

$\square$

Remark 2.5.8. *Let (X, d) be a quasi-median space and δ_0 be the optimal constant such that X is a δ_0-median space with **(δ_0-Med2)** constant denoted by C_0. From the proof of Proposition 2.5.6, when $\delta \geq \delta_0$ varies, the function $\delta \mapsto C_\delta\delta$, where C_δ is the **(δ-Med2)** constant of X (seen as a δ-median space), is a linear function of δ. In fact, we have :*

$$C_\delta\delta = 3\delta + (20C_0 + \frac{51}{2})\delta_0.$$

*In particular, if (X, d) is 0-median, then, for $\delta \geq 0$, X is δ-median with **(δ-Med2)** constant $C_\delta = 3$.*

Proof of Theorem 7. Assume (X,d) is a δ-median space with **(δ-Med2)** constant denoted by C. Let $\delta' \geq \delta$ and let $a,b,c \in X$.

By **(δ-Med1)**, there exists $x \in [a,b]$,$y \in [b,c]$ and $z \in [c,a]$ $\frac{\delta}{2}$-close points of $\Delta(a,b,c)$. A fortiori, x,y,z are pairwise $\frac{\delta'}{2}$-close, then $\Delta(a,b,c)$ is $\frac{\delta'}{2}$-thin. It follows that (X,d) is $L'_{\delta'}$.

By Corollary 2.5.7, there exists $C' \geq 0$ which does not depend on a,b,c such that $diam(M_{\delta'}(a,b,c)) \leq C'\delta'$, and hence, (X,d) satisfies **(δ'-Med2)**. $\square$

2.6 Stability properties

2.6.1 Direct product

Proposition 2.6.1. *Let (X_1,d_1), (X_2,d_2) be, respectively, δ_1-median and δ_2-median spaces. Then $(X_1 \times X_2, d_1+d_2)$ is a δ-median space where $\delta = \delta_1 + \delta_2$.*

Proof. By Theorem 7, X_1 and X_2 are δ-median spaces for $\delta := \delta_1+\delta_2$. Denote C_1 and C_2, the **(δ-Med2)** constants of X_1 and X_2 respectively, and set $C := C_1 + C_2$. Let us show that (X,d) is δ-median where $X = X_1 \times X_2$, $d = d_1 + d_2$.

Let $a = (a_1,a_2)$, $b = (b_1,b_2)$, $c = (b_1,b_2) \in X$.

(δ-Med1) : For $i = 1,2$, consider the $\frac{\delta_i}{2}$-thin full triangle $\Delta_i := \Delta_f(a_i,b_i,c_i)$, and let $p_i \in [a_i,b_i]$, $q_i \in [b_i,c_i]$ and $r_i \in [c_i,a_i]$ be some $\frac{\delta_i}{2}$-close points of Δ_i. Then, the pairs (p_1,p_2), (q_1,q_2) et (r_1,r_2) are pairwise $\frac{\delta}{2}$-close with respect to $d = d_1+d_2$ and, by definition, $(p_1,p_2) \in [a,b]$, $(q_1,q_2) \in [b,c]$ and $(r_1,r_2) \in [c,a]$. It follows that the full triangle $\Delta_f(a,b,c)$ is $\frac{\delta}{2}$-thin in (X,d).

(δ-Med2) : Let $m = (m_1,m_2) \in M_\delta(a,b,c)$. Since $m \in [a,b]_\delta$, we have :

$$d_1(a_1,m_1) + d_1(b_1,m_1) + d_2(a_2,m_2) + d_2(b_2,m_2) \leqslant d_1(a_1,b_1) + d_2(a_2,b_2) + \delta.$$

Moreover, by triangular inequality, we have $d(a_i,b_i) \leqslant d_i(a_i,m_i) + d_i(b_i,m_i)$, for $i = 1,2$, and hence :

$$d_i(a_i,m_i) + d_i(b_i,m_i) \leqslant d(a_i,b_i) + \delta.$$

By expressing that $m \in [b,c]_\delta$ and $m \in [c,a]_\delta$, analoguous inequalities hold for the pairs (b_i,c_i),(c_i,a_i), thus $m_i \in M_\delta(a_i,b_i,c_i)$ for $i = 1,2$. As a consequence, if $m = (m_1,m_2)$, $m' = (m'_1,m'_2) \in M_\delta(a,b,c)$, then, for $i = 1,2$, by **(δ-Med2)** for the δ-median space (X_i,d_i),

$$d(m_i,m'_i) \leq C_i\delta.$$

We have $diam(M_\delta(a,b,c)) = diam(M_\delta(a_1,b_1,c_1)) + diam(M_\delta(a_2,b_2,c_2)) \leqslant C_1\delta + C_2\delta$ and hence, it follows that :

$$diam(M_\delta(a,b,c)) \leq C\delta.$$

$\square$

Example 2.6.2. *A finite product of Gromov hyperbolic spaces is a quasi-median space with respect to the sum of the metrics.*

2.6.2 Free product

In this part, we consider the notion of free product of metric spaces defined by Dreesen in [Dre11] Remark 2.6 and developed in Section 1.6 and we give a proof of Theorem 8.

We recall here the basic definitions of free product of spaces and we give more details on the natural metric on the free product induced by metrics on each factors.

Given a set of symbols S, we denote by $\mathcal{M}(S)$ the set of words in S i.e. $\mathcal{M}(S) = \{s_1...s_n \mid s_i \in S,\ n \in \mathbb{N}\}$.

Definition (recall of Definition 1.6.1)**.** *Let S, S' be sets. An* alternating word *in S and S' is a word $w_1w_2...w_n \in \mathcal{M}(S \sqcup S')$ such that, for $i = 1, ..., n-1$, either $w_i \in S$ and $w_{i+1} \in S'$, or $w_i \in S'$ and $w_{i+1} \in S$.*
We denote $Alt(S,S')$ the set of all alternating words in S and S'.

Definition (recall of Definition 1.6.2)**.** *Let X, Y be non-empty sets and fix $x_0 \in X$, $y_0 \in Y$. The* free product *of X and Y on the basepoints x_0, y_0 is the set :*

$$X \underset{x_0 \sim y_0}{*} Y = Alt(X \smallsetminus \{x_0\}, Y \smallsetminus \{y_0\}).$$

When there is no ambiguity on the fixed points, we simply denote the free product of X and Y by $X * Y$.

Subsequently, we denote by e the empty word of $X * Y$.

1. Word paths and inner triples

For the next definition, we need the notion of left-cosets in $X*Y$ defined in Part 1.6.1, Definition 1.6.5.

Definition 2.6.3 (Pair decomposition)**.** *Let w_1, w_2 be alternating words of $X*Y$; w_1 and w_2 can be written in the following form :*

$$w_1 = w_c d_1 w_1',$$
$$w_2 = w_c d_2 w_2',$$

where :

- *$w_c \in X*Y$ is the largest common beginning word of w_1 and w_2 (w_c can possibly be the empty word e), that is, w_c is the maximal element of $Sub(w_1) \cap Sub(w_2)$ for the order relation $\leq$ (see Remark 1.6.4) ;*
- *d_1, d_2 are the "first different" letters after w_c of w_1 and w_2 respectively, that is :*
  · when $w_c \neq e$, we set (for $Z = X$ or Y as appropriate) :

$$d_1 = \pi_{w_c Z}(w_1) \text{ and } d_2 = \pi_{w_c Z}(w_2);$$

  · when $w_c = e$, if the first letters of w_1, w_2 belong to the same set $Z = X$ or Y (this includes w_1 or w_2 equal to e) we set :

$$d_1 = \pi_{eZ}(w_1) \text{ and } d_2 = \pi_{eZ}(w_2);$$

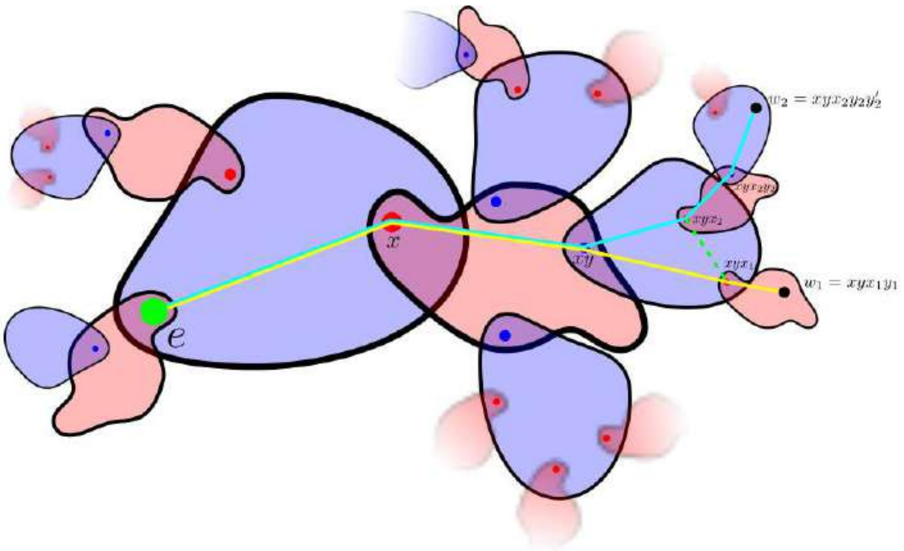
  otherwise we choose to define d_1 and d_2 as the eX-projection of w_1 and w_2 :

$$d_1 = \pi_{eX}(w_1) \text{ and } d_2 = \pi_{eX}(w_2);$$

- *$w_1', w_2' \in X*Y$ are the tails of w_1, w_2 respectively.*

This decomposition is called the pair decomposition *of w_1, w_2.*

Remark 2.6.4. *Let $w_1, w_2 \in X*Y$.*
- In the pair decomposition of w_1, w_2, d_1, d_2 are defined in such a way that they belong to the same set X or Y. Notice that d_1 or d_2 can possibly be equal to x_0 or y_0.
- We have $d_1 = d_2$ if, and only if, $w_1 = w_2$.



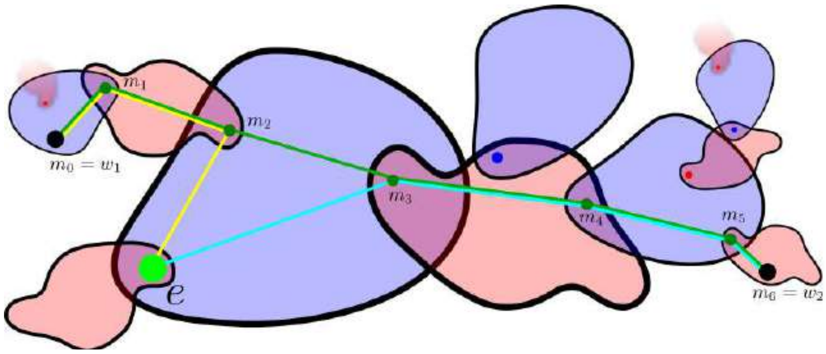
The pair decomposition of $w_1 = xyx_1y_1$ and $w_2 = xyx_2y_2x'_1y'_2$ is given by :

$$w_c = xy\,;$$
$$d_1 = x_1,\ d_2 = x_2\,;$$
$$w'_1 = y_1,\ w'_2 = y_2x'_1y'_2.$$

Definition 2.6.5. *Let $w_1, w_2 \in X * Y$ and consider the pair decomposition $w_1 = w_cd_1w'_1$ and $w_2 = w_cd_2w'_2$ with $w'_1 = w'_{1,1}...w'_{1,n}$ and $w'_2 = w'_{2,1}...w'_{2,m}$. We define the family $(m_0(w_1,w_2),...,m_{n+m+1}(w_1,w_2))$ called the* word path from w_1 to w_2 *by :*

$$\begin{aligned}
m_0(w_1,w_2) &= w_cd_1w'_{1,1}...w'_{1,n} = w_1,\\
m_i(w_1,w_2) &= w_cd_1w'_{1,1}...w'_{1,n-i} \text{ for } i = 1,...,n-1,\\
m_n(w_1,w_2) &= w_cd_1,\\
m_{n+1}(w_1,w_2) &= w_cd_2,\\
m_{n+i+1}(w_1,w_2) &= w_cd_2w'_{2,1}...w'_{2,m-i}, \text{ for } i = 1,...,m-1,\\
m_{n+m+1}(w_1,w_2) &= w_cd_2w'_{2,1}...w'_{2,m} = w_2.
\end{aligned}$$

In the degenerated cases where d_1 or d_2 is x_0, y_0, we consider x_0 and y_0 as the empty letter e.g. if $w_1 = xy$ and $w_2 = yx$, the word path from w_1 to w_2 is (xy, x, e, y, yx) ; or if $w_1 = w_2$, the word path is (w_1, w_2).



The word path from w_1 to w_2.

Remark 2.6.6. - *The word path between w_1 and w_2 does not depend on the choice for d_1, d_2 we made in Definition 2.6.3 in the case $w_c = e$: if we had chosen the eY-projection instead of the eX-projection, the word path between two words would still be the same.*
- Notice that if $(m_i(w_1, w_2))_{i=0,\dots,k}$ is the word path from w_1 to w_2, then $(m_{k-i}(w_1, w_2))_{i=0,\dots,k}$ is the word path from w_2 to w_1.

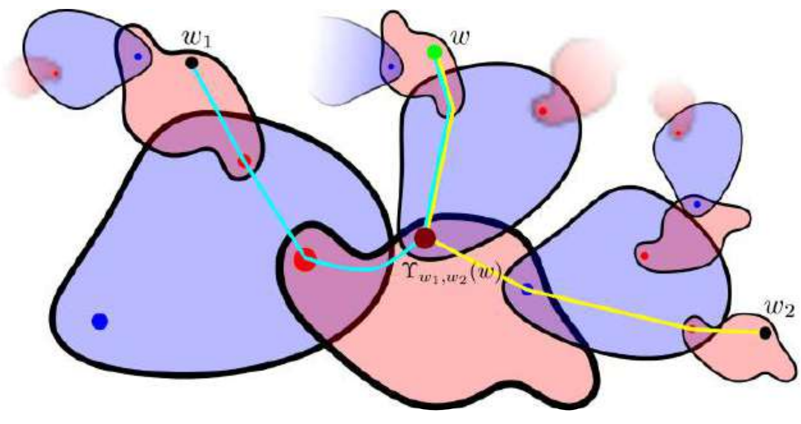
Definition 2.6.7 (Word path divergence)**.** *Let $w_1, w_2 \in X * Y$. Consider $w \in X * Y$ and word paths $(m_0(w, w_1), \dots, m_k(w, w_1))$ and $(m_0(w, w_2), \dots, m_l(w, w_2))$ from, respectively w to w_1, and w to w_2. We set :*

$$j_{w_1,w_2}(w) = \max\{j \leq \min\{k, l\} \mid m_j(w, w_1) = m_j(w, w_2)\},$$

and

$$\Upsilon_{w_1,w_2}(w) = m_{j_{w_1,w_2}(w)}(w, w_1)(= m_{j_{w_1,w_2}(w)}(w, w_2)).$$

*The map $\Upsilon_{w_1,w_2} : X * Y \to X * Y$ by is called the w_1, w_2-*divergence.



The w_1, w_2-divergence $\Upsilon_{w_1,w_2}(w)$ of a word $w \in X * Y$.

Remark 2.6.8. — *For every $j \leq j_{w_1,w_2}(w)$, we have $m_j(w,w_1) = m_j(w,w_2)$.*

— *The w_1,w_2-divergence $\Upsilon_{w_1,w_2}(w)$ of a word $w \in X * Y$ is the last common point of the word paths from w to w_1 and from w to w_2 before they split.*

Definition 2.6.9 (Inner triple)**.** *Let $w_1, w_2, w_3 \in X * Y$.*
The triple $\Upsilon_{w_1,w_2}(w_3), \Upsilon_{w_2,w_3}(w_1), \Upsilon_{w_3,w_1}(w_2)$ is called the inner triple of w_1, w_2, w_3.

The next proposition states that the three word paths joining w_1, w_2 and w_3 pairwise split in the same left-coset in $X * Y$.

Proposition 2.6.10. *Let $w_1, w_2, w_3 \in X * Y$. Then there exists a left-coset wZ such that the alternating words of the inner triple of w_1, w_2, w_3 belong to wZ.*

Proof. Consider the inner triple $\Upsilon_{w_1,w_2}(w_3), \Upsilon_{w_2,w_3}(w_1), \Upsilon_{w_3,w_1}(w_2)$ of w_1, w_2, w_3. The statement is clearly true when at least two words among w_1, w_2 and w_3 are equal. Let us assume w_1, w_2, w_3 are pairwise distinct.
The pair decompositions for w_1, w_2, w_3 are :

$$\begin{array}{lll} w_1 = w_c d_1 v_1, & w_2 = w'_c d'_2 v'_2, & w_3 = w''_c d''_3 v''_3, \\ w_2 = w_c d_2 v_2; & w_3 = w'_c d'_3 v'_3; & w_1 = w''_c d''_1 v''_1. \end{array}$$

Up to permute w_1, w_2 and w_3, we can assume that $w'_c, w''_c \in \mathrm{Sub}(w_c)$ (i.e. w_c is the word of largest length among w_c, w'_c and w''_c). Notice that it implies $w'_c = w''_c$. In fact, we have :

$$w''_c \in \mathrm{Sub}(w_c) \cap \mathrm{Sub}(w_3) \subset \mathrm{Sub}(w_2) \cap \mathrm{Sub}(w_3) = \mathrm{Sub}(w'_c),$$

and

$$w'_c \in \mathrm{Sub}(w_c) \cap \mathrm{Sub}(w_3) \subset \mathrm{Sub}(w_1) \cap \mathrm{Sub}(w_3) = \mathrm{Sub}(w''_c);$$

and hence, $w''_c \leq w'_c$ and $w'_c \leq w''_c$.
Thus, there exists $u = u_1 ... u_k \in X * Y$ such that $w_c = w'_c u$ and moreover, $d'_3 = d''_3$ and $v'_3 = v''_3$; hence, we can reformulate the pair decompositions as follows :

$$\begin{array}{lll} w_1 = w'_c u d_1 v_1, & w_2 = w'_c d'_2 v'_2, & w_3 = w'_c d'_3 v'_3, \\ w_2 = w'_c u d_2 v_2; & w_3 = w'_c d'_3 v'_3; & w_1 = w'_c d''_1 v''_1. \end{array}$$

Let us show that $\Upsilon_{w_1,w_2}(w_3), \Upsilon_{w_2,w_3}(w_1), \Upsilon_{w_3,w_1}(w_2)$ belongs to a left-coset associated with w_c.

First case : $u = u_1...u_k \neq e$ (i.e. $w'_c \in \mathrm{Sub}^*(w_c)$).
It follows that $d'_2 = u_1 = d''_1$ in the pair decompositions of w_2, w_3 and w_3, w_1, and hence, the elements $w'_c u_1, ..., w'_c u_1...u_k = w_c$ are common to the word paths from w_3 to w_1 and from w_3 to w_2. Moreover, since $u \neq e$, $w_c \neq e$ and then $d_1 \neq d_2$; thus the two word paths w_3 to w_1 and from w_3 to w_2 splits just after w_c as $w_c d_1 \neq w_c d_2$. It follows that :

$$\Upsilon_{w_1,w_2}(w_3) = w_c.$$

Now, $w_c d_1$ belong to the word paths from w_1 to w_2 (by definition) and from w_1 to w_3 and notice that w_c does not belong to the word path from w_1 to w_2. Hence, we have $\Upsilon_{w_2,w_3}(w_1) = w_c d_1$ and by the same arguments for $w_c d_1$ for the word paths from w_2, $\Upsilon_{w_3,w_1}(w_2) = w_c d_2$. It follows that :

$$\Upsilon_{w_2,w_3}(w_1), \Upsilon_{w_3,w_1}(w_2), \Upsilon_{w_1,w_2}(w_3) \in w_c Z.$$

Second case : $u = e$ (i.e. $w'_c = w_c$).
In this case, the pair decompositions become :

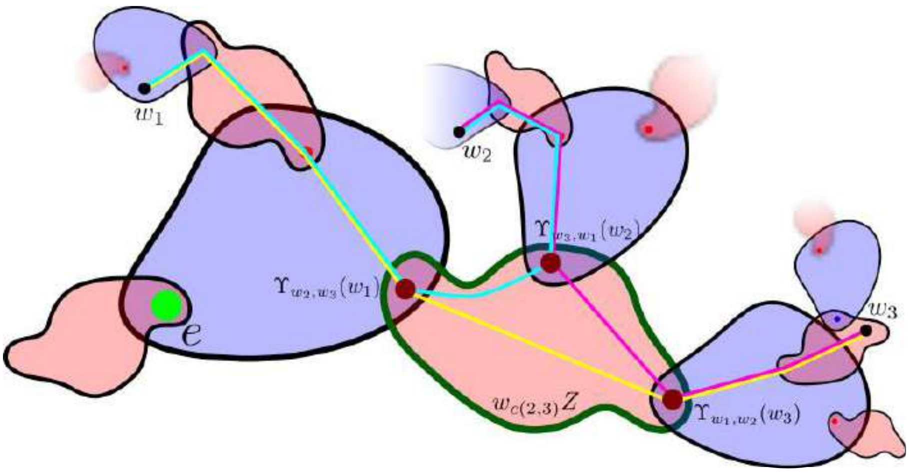
$$\begin{array}{lll} w_1 = w_c d_1 v_1, & w_2 = w_c d_2 v_2, & w_3 = w_c d'_3 v'_3, \\ w_2 = w_c d_2 v_2; & w_3 = w_c d'_3 v'_3; & w_1 = w_c d_1 v_1. \end{array}$$

- If $w_c \neq e$ and $w_c \in E_Z$, it follows that $\Upsilon_{w_2,w_3}(w_1) = w_c d_1, \Upsilon_{w_3,w_1}(w_2) = w_c d_2$ and $\Upsilon_{w_1,w_2}(w_3) = w_c d'_3$ belong to $w_c Z$.
- If $w_c = e$, then two words among w_1, w_2, w_3 starts with a letter in the same set $Z = X$ or Y, and the third one starts with a letter in the other set. We can assume without loss of generality that w_2, w_3 starts with a letter in the same set Z (but necessarily different letters since $w_2 \neq w_3$ and $w_c = e$). Hence, we have $w_1 = tw'_1$, $w_2 = zw'_2$, $w_3 = z'w'_3$ with $w'_i \in X * Y$, $z, z' \in Z$ and $t \notin Z$. Hence :

$$\Upsilon_{w_2,w_3}(w_1) = e,\ \Upsilon_{w_3,w_1}(w_2) = z \text{ and } \Upsilon_{w_1,w_2}(w_3) = z'.$$

and then, $\Upsilon_{w_2,w_3}(w_1) = e, \Upsilon_{w_3,w_1}(w_2), \Upsilon_{w_1,w_2}(w_3) \in eZ = w_c Z$.

In every case, the inner triple of w_1, w_2, w_3 belong to a left-coset associated with the largest subword in the pair decompositions of w_1, w_2, w_3. $\square$



The inner triangle of w_1, w_2, w_3.

2. Free product of metric spaces

Definition 2.6.11. *Let (X, d_X) and (Y, d_Y) be non-empty metric spaces and $x_0 \in X$, $y_0 \in Y$ be basepoints.*

— *The* length *of an alternating word $w = w_1...w_n \in X * Y$ with respect to the metrics d_X and d_Y, is defined by :*

$$l_{X*Y}(w) = \sum_{i=1}^{n} d_{X,Y}(w_i, z_0),$$

where $d_{X,Y}$ stands for d_X or d_Y and z_0 stands for x_0 or y_0 as appropriate.

— *The distance $d_{X*Y}(w_1, w_2)$ for $w_1, w_2 \in X * Y$ written $w_1 = w_c d_1 w_1'$ and $w_2 = w_c d_2 w_2'$ is given as follows :*

$$d_{X*Y}(w_1, w_2) = d_{X,Y}(d_1, d_2) + l_{X*Y}(w_1') + l_{X*Y}(w_2'),$$

and the metric d_{X*Y} is called the free product metric *on the free product of the metric spaces (X, d_X) and (Y, d_Y).*

Lemma 2.6.12. *Let (X, d_X) and (Y, d_Y) be non-empty metric spaces and $x_0 \in X$, $y_0 \in Y$ be basepoints. For all $w, w' \in X * Y$, the word path $(m_0(w, w'), ..., m_k(w, w'))$ from w to*

w' is a geodesic path for d_{X*Y} i.e.

$$\sum_{i=0}^{k-1} d_{X*Y}(m_i(w,w'), m_{i+1}(w,w')) = d_{X*Y}(w,w').$$

Proof. Let $w = w_c du$, $w' = w_c d'v$ be the pair decomposition of w, w' with $u = u_1...u_n$ and $v = v_1...v_m$ in $X * Y$. Then the word path $(m_0, ..., m_{n+m+1})$ from w_1 to w_2 is given by :

- $m_i = w_c du_{n-i}$ for $i = 0, ..., n$ and,
- $m_{n+i+1} = w_c d'v_i$ for $i = 0, ..., m$.

Hence, we have :

- for $i = 0, ..., n-1$,
$$d_{X*Y}(m_i, m_{i+1}) = d_{X,Y}(u_{n-i}, z_0);$$
- for $i = n$,
$$d_{X*Y}(m_n, m_{n+1}) = d_{X,Y}(d_1, d_2); \text{ and}$$
- for $i = 1, ..., m$,
$$d_{X*Y}(m_{n+i}, m_{n+i+1}) = d_{X,Y}(v_i, z_0).$$

It follows that :

$$\begin{aligned}
\textstyle\sum_{i=0}^{n+m} d_{X*Y}(m_i, m_{i+1}) &= \textstyle\sum_{i=0}^{n-1} d_{X*Y}(m_i, m_{i+1}) + d_{X*Y}(m_n, m_{n+1}) + \sum_{i=1}^{m} d_{X*Y}(m_{n+i}, m_{n+i+1}),\\
&= \textstyle\sum_{i=0}^{n-1} d_{X,Y}(u_{n-i}, z_0) + d_{X,Y}(d_1, d_2) + \sum_{i=1}^{m} d_{X,Y}(v_i, z_0),\\
&= l_{X*Y}(u) + d_{X,Y}(d_1, d_2) + l_{X*Y}(v),\\
\textstyle\sum_{i=0}^{n+m} d_{X*Y}(m_i, m_{i+1}) &= d_{X*Y}(w, w').
\end{aligned}$$

□

Proposition 2.6.13. *Let wZ be a left-coset of $X * Y$ where $w \in X * Y$ and $Z = X$ or Y as appropriate and π_{wZ} be the wZ-projection on Z. Then the restriction :*

$$\pi_{wZ} : (wZ, d_{X*Y}) \to (Z, d_Z),$$

is a bijective isometry whose inverse map π_{wZ}^{-1} is given by $\pi_{wZ}^{-1}(z) = wz$, for $z \in Z$.

Proof. Let $wz, wz' \in wZ$ with $z \neq z' \in Z$. We have :

$$\pi_{wZ}(wz) = z \text{ and } \pi_{wZ}(wz') = z'.$$

Consider the pair decomposition of wz, wz' :

$$wz = w_c du; \;\; wz' = w_c d' u'.$$

Then $w_c = w$, $d = z$, $d' = z'$ and $u = u' = e$. It follows that :

$$d_{X*Y}(wz, wz') = d_Z(z, z') = d_Z(\pi_{wZ}(wz), \pi_{wZ}(wz')).$$

Hence, the restriction π_{wZ} to wZ is a bijective isometry and it is clear that $\pi_{wZ}^{-1} : z \mapsto wz$ is its inverse map. $\square$

Corollary 2.6.14. *Let $\alpha \geq 0$. Let wZ be the left-coset and $u, u', u'' \in wZ$. Then*

$$u \in [u', u'']_a \Leftrightarrow \pi_{wZ}(u) \in [\pi_{wZ}(u'), \pi_{wZ}(u'')]_a.$$

Proof. It is an immediate consequence of the previous proposition, in fact :

$$d_{X*Y}(u', u) + d_{X*Y}(u, u'') = d_Z(\pi_{wZ}(u'), \pi_{wZ}(u)) + d_Z(\pi_{wZ}(u), \pi_{wZ}(u'')).$$

$\square$

Proposition 2.6.15. *Let (X, d_X), (Y, d_Y) be non-empty metric spaces and $x_0 \in X$, $y_0 \in Y$ be basepoints. If d_X and d_Y are geodesic metrics, then d_{X*Y} is a geodesic metric.*

Proof. Let $w, w' \in X * Y$, $w = w_c du$, $w' = w_c d' v$ be the pair decomposition of w, w' with $u = u_1...u_n$ and $v = v_1...v_m$ in $X * Y$ and $(m_0, ..., m_{n+m+1})$ be the word path from w to w'. Subsequently, Z stands for X or Y and z_0 for x_0 or y_0 as appropriate. As (X, d_X) and (Y, d_Y) are geodesic,

— for $i = 1, ..., n$, there exists a geodesic arc γ_{u_i} in Z from u_i to z_0 ;

— there exists a geodesic arc $\gamma_{d,d'}$ in Z from d to d' ;

— for $i = 1, ..., m$, there exists a geodesic arc γ_{v_i} in Z from z_0 to v_i.

Hence, we can define the following family $(\gamma_i)_{i=0,...,n+m}$ of arcs in $X * Y$

— for $i = 0, ..., n-1$, $\gamma_i(t) = w_c d u_1 ... u_{n-i-1} \gamma_{u_{n-i}}(t)$ for $t \in [0, d_Z(u_{n-i}, z_0)]$;

— $\gamma_n(t) = w_c \gamma_{d,d'}(t)$ for $t \in [0, d_Z(d, d')]$;

— for $i = n+1, ..., n+m$, $\gamma_i(t) = w_c d' v_1 ... v_{i-n-1} \gamma_{v_{i-n}}(t)$ for $t \in [0, d_Z(v_{i-n}, z_0)]$.

The family (γ_i) is a family of geodesic arcs in $X * Y$ since the γ_{u_i}'s, $\gamma_{d,d'}$ and the γ_{v_i}'s are geodesic arcs in Z, and moreover, for each i, γ_i is a geodesic arc from m_i to m_{i+1}. By Lemma 2.6.12, $(m_0, ..., m_{n+m+1})$ is a geodesic path for d_{X*Y}, hence, by Corollary 2.2.8, there exists a geodesic arc from $m_0 = w$ to $m_{n+m+1} = w'$. $\square$

Remark 2.6.16. *If G, H are groups, as sets, the usual free product of groups $G * H$ and the free product $G \underset{e_G \sim e_H}{*} H$ coincide. Moreover, if G, H are finitely generated and endowed with the word metric relative to generating sets S_G and S_H respectively, then the word metric on $G * H$ relative to the generating set $S_G \sqcup S_H$ is the same as the free product metric on the free product of the metric spaces (G, d_{S_G}) and (H, d_{S_H}).*

3. Free product of quasi-median spaces

Let (X, d_X) and (Y, d_Y) be non-empty metric spaces, $x_0 \in X$, $y_0 \in Y$ be basepoints and $(X * Y, d_{X*Y})$ be the free product of the metric spaces (X, d_X) and (Y, d_Y).

The metric properties of a geodesic triangle $\Delta(w_1, w_2, w_3)$ in the free product $X * Y$ lie in its "inner" geodesic triangle $\Delta(\Upsilon_{w_2,w_3}(w_1), \Upsilon_{w_3,w_1}(w_2), \Upsilon_{w_1,w_2}(w_3))$ which can be seen as a geodesic triangle in X or Y by Proposition 2.6.10. Hence, metric triangle properties like $CAT(\kappa)$ property or Gromov hyperbolicity are stable by free product of metric spaces. As we shall see, the same goes for the δ-median property.

Proposition 2.6.17. *Let $\delta \geq 0$. For all $w_1, w_2, w_3 \in X * Y$,*

$$M_\delta(w_1, w_2, w_3) = M_\delta(\Upsilon_{w_2,w_3}(w_1), \Upsilon_{w_3,w_1}(w_2), \Upsilon_{w_1,w_2}(w_3)),$$

and, moreover, if wZ is the left-coset containing $\Upsilon_{w_2,w_3}(w_1), \Upsilon_{w_3,w_1}(w_2), \Upsilon_{w_1,w_2}(w_3)$ given by Proposition 2.6.10, we have :

$$M_\delta(w_1, w_2, w_3) \subset V_{\frac{\delta}{2}}\left(M_\delta(\Upsilon_{w_2,w_3}(w_1), \Upsilon_{w_3,w_1}(w_2), \Upsilon_{w_1,w_2}(w_3)) \cap wZ\right).$$

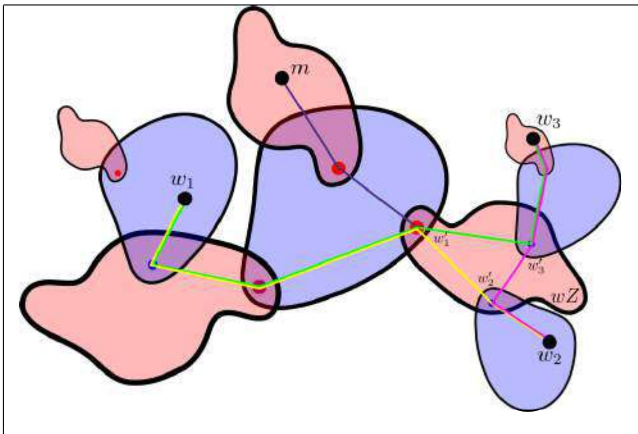


Figure 1.

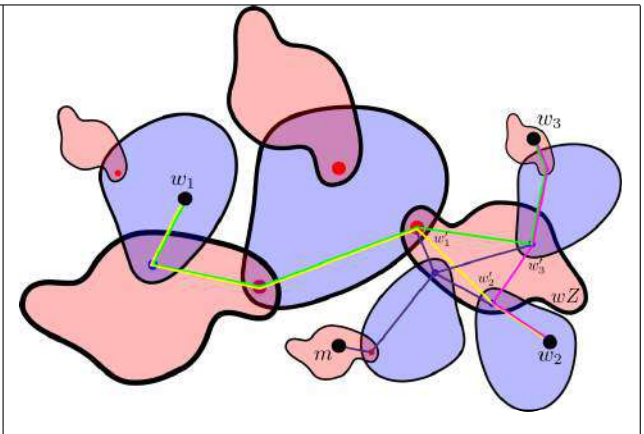


Figure 2.

Proof. Let $w_1, w_2, w_3 \in X$. We set $w'_1 = \Upsilon_{w_2,w_3}(w_1)$, $w'_2 = \Upsilon_{w_3,w_1}(w_2)$ and $w'_3 = \Upsilon_{w_1,w_2}(w_3)$, and let wZ be the left-coset containing w'_1, w'_2, w'_3. Let us show that $M_\delta(w_1, w_2, w_3) \subset M_\delta(w'_1, w'_2, w'_3)$.
Let $m \in M_\delta(w_1, w_2, w_3) = [w_1, w_2]_\delta \cap [w_2, w_3]_\delta \cap [w_3, w_1]_\delta$.

First case : $\pi_{wZ}(m) \in \{\pi_{wZ}(w'_1), \pi_{wZ}(w'_2), \pi_{wZ}(w'_3)\}$ (Figure 1.).
Up to permute w_1, w_2, w_3, we can assume that $\pi_{wZ}(m) = \pi_{wZ}(w'_1)$. Then (w_2, w'_2, w'_1, m) and (w_3, w'_3, w'_1, m) are geodesic paths and hence,

$$\begin{aligned}
d_{X*Y}(w'_2, m) + d_{X*Y}(m, w'_3) &= d_{X*Y}(w_2, m) - d_{X*Y}(w_2, w'_2) + d_{X*Y}(m, w_3) - d_{X*Y}(w'_3, w_3) \\
&\leq d(w_2, w_3) - (d_{X*Y}(w_2, w'_2) + d_{X*Y}(w'_3, w_3)) + \delta \\
d_{X*Y}(w'_2, m) + d_{X*Y}(m, w'_3) &\leq d_{X*Y}(w'_2, w'_3) + \delta,
\end{aligned}$$

which means that $m \in [w'_2, w'_3]_\delta$. Moreover, we have :

$$\begin{aligned}
d_{X*Y}(w'_2, w'_1) + d_{X*Y}(w'_1, m) + d_{X*Y}(m, w'_1) + d_{X*Y}(w'_1, w'_3) &= d_{X*Y}(w'_2, m) + d_{X*Y}(m, w'_3) \\
&\leq d_{X*Y}(w'_2, w'_3) + \delta \\
&\leq d_{X*Y}(w'_2, w'_1) + d_{X*Y}(w'_1, w'_3) + \delta,
\end{aligned}$$

and then, $d_{X*Y}(w'_1, m) \leq \frac{\delta}{2}$. By Lemma 2.2.4, it follows that $m \in [w'_1, w'_2]_\delta \cap [w'_3, w'_1]_\delta$. As a conclusion, $m \in M_\delta(w'_1, w'_2, w'_3)$.

Second case : $\pi_{wZ}(m) \in Z \smallsetminus \{\pi_{wZ}(w'_1), \pi_{wZ}(w'_2), \pi_{wZ}(w'_3)\}$ (Figure 2.).
Then (w_1, w'_1, m), (w_2, w'_2, m), (w_3, w'_3, m) are geodesic paths, and we have :

$$\begin{aligned} d_{X*Y}(w'_1, m) + d_{X*Y}(m, w'_2) &= d_{X*Y}(w_1, m) - d_{X*Y}(w_1, w'_1) + d_{X*Y}(m, w_2) - d_{X*Y}(w'_2, w_2) \\ &\leq d(w_1, w_2) - (d_{X*Y}(w_1, w'_1) + d_{X*Y}(w'_2, w_2)) + \delta \\ d_{X*Y}(w'_1, m) + d_{X*Y}(m, w'_2) &\leq d_{X*Y}(w'_1, w'_2) + \delta. \end{aligned}$$

Hence, $m \in [w'_1, w'_2]_\delta$, and by similar computations, $m \in [w'_2, w'_3]_\delta$ and $m \in [w'_3, w'_1]_\delta$. Thus $m \in M_\delta(w'_1, w'_2, w'_3)$.

It follows that we have the inclusion $M_\delta(w_1, w_2, w_3) \subset M_\delta(w'_1, w'_2, w'_3)$ and moreover, the reverse inclusion is clear since (w_i, w'_i, w'_j, w_j) is a geodesic path for $i, j = 1, 2, 3$ with $i \neq j$.

Let us show the second statement. Let $m \in M_\delta(w'_1, w'_2, w'_3)$ and $z = \pi_{wZ}(m)$. Then (w'_1, wz, m), (w'_2, wz, m) and (w'_3, wz, m) are geodesic paths and we have :

$$d_{X*Y}(w'_1, wz) + d_{X*Y}(wz, w'_2) \leq d_{X*Y}(w'_1, m) + d_{X*Y}(m, w'_2) \leq d_{X*Y}(w'_1, w'_2) + \delta.$$

Similar arguments holds for the pairs w'_2, w'_3 and w'_3, w'_1 and hence, $wz \in M_\delta(w'_1, w'_2, w'_3) \cap wZ$.

Moreover, $2d(wz, m) \leq d_{X*Y}(w'_1, w'_2) - (d_{X*Y}(w'_1, wz) + d_{X*Y}(wz, w'_2)) + \delta \leq \delta$, thus m and wz are $\frac{\delta}{2}$-close. Finally, we have :

$$M_\delta(w_1, w_2, w_3) = M_\delta(w'_1, w'_2, w'_3) \subset V_{\frac{\delta}{2}}\left(M_\delta(w'_1, w'_2, w'_3) \cap wZ\right).$$

□

Corollary 2.6.18. *Let $\delta \geq 0$. Assume there exists $K \geq 0$ such that, for all $x, x', x'' \in X$ and for all $y, y', y'' \in Y$, $diam(M_\delta(x, x', x'')) \leq K$ and $diam(M_\delta(y, y', y'')) \leq K$. Then for all $w_1, w_2, w_3 \in X * Y$,*

$$diam(M_\delta(w_1, w_2, w_3)) \leq K + \delta.$$

Proof. Let $w_1, w_2, w_3 \in X * Y$, $w'_1 = \Upsilon_{w_2,w_3}(w_1)$, $w'_2 = \Upsilon_{w_3,w_1}(w_2)$, $w'_3 = \Upsilon_{w_1,w_2}(w_3)$ be the inner triple of w_1, w_2, w_3 and wZ be the left-coset containing w'_1, w'_2, w'_3. Let $m, m' \in M_\delta(w_1, w_2, w_3)$. Then, by the previous proposition, there exists $m_0, m'_0 \in M_\delta(w'_1, w'_2, w'_3) \cap wZ$ such that $d_{X*Y}(m, m_0) \leq \frac{\delta}{2}$ and $d_{X*Y}(m, m_0) \leq \frac{\delta}{2}$ and notice that, by Corollary 2.6.14,

$$\pi_{wZ}(m_0), \pi_{wZ}(m'_0) \in M_\delta(\pi_{wZ}(w'_1), \pi_{wZ}(w'_2), \pi_{wZ}(w'_3)) \subset Z,$$

and hence, by hypothesis, $d_Z(\pi_{wZ}(m_0), \pi_{wZ}(m_0')) \leq K$.
It follows that, by Proposition 2.6.13 $d_{X*Y}(m_0, m_0') = d_Z(\pi_{wZ}(m_0), \pi_{wZ}(m_0')) \leq K$; and finally,

$$d_{X*Y}(m, m') \leq d_{X*Y}(m, m_0) + d_{X*Y}(m_0, m_0') + d_{X*Y}(m_0', m') \leq K + \delta.$$

□

Proposition 2.6.19. *Let $\delta, \delta' \geq 0$.*

1. *If (X, d_X) is L_δ and (Y, d_Y) is L_δ, then $(X * Y, d_{X*Y})$ is $L_{\max(\delta,\delta')}$.*
2. *If (X, d_X) is L'_δ and (Y, d_Y) is L'_δ, then $(X * Y, d_{X*Y})$ is $L'_{\max(\delta,\delta')}$.*

Proof. Let $\alpha = \max(\delta, \delta')$. Let $w_1, w_2, w_3 \in X$ and $w_1' = \Upsilon_{w_2,w_3}(w_1), w_2' = \Upsilon_{w_3,w_1}(w_2), w_3' = \Upsilon_{w_1,w_2}(w_3)$ be the inner triple of w_1, w_2, w_3. Then, by Proposition 2.6.10, there exists a left-coset wZ such that $w_1', w_2', w_3' \in wZ$. We set :

$$z_1 = \pi_{wZ}(w_1'),\ z_2 = \pi_{wZ}(w_2') \text{ and } z_3 = \pi_{wZ}(w_3').$$

1. Assume that (X, d_X) is L_δ and (Y, d_Y) is $L_{\delta'}$. Then, by Remark 2.3.8, for $Z = X$ or Y, (Z, d_Z) is L_α and hence, there exists $z \in M_\alpha(z_1, z_2, z_3)$. It follows, by Corollary 2.6.14 and Proposition 2.6.17, that

$$wz \in M_\alpha(w_1', w_2', w_3') = M_\alpha(w_1, w_2, w_3).$$

This shows that $X * Y$ is L_α.

2. Assume that (X, d_X) is L'_δ and (Y, d_Y) is $L'_{\delta'}$. Then, by Remark 2.3.8, for $Z = X$ or Y, (Z, d_Z) is L'_α. Let $z_1' \in [z_1, z_2], z_2' \in [z_2, z_3], z_3' \in [z_3, z_1]$ be $\frac{\alpha}{2}$-inner points of $\Delta_f(z_1, z_2, z_3)$. We have :

— $wz_1' \in [w_1', w_2']$, $wz_2' \in [w_2', w_3']$ and $wz_3' \in [w_3', w_1']$ by Corollary 2.6.14 ;

— $d_{X*Y}(wz_i', wz_j') = d_Z(z_i, z_j) \leq \frac{\delta}{2}$ by Proposition 2.6.13.

Thus, wz_1', wz_2', wz_3' are $\frac{\alpha}{2}$-inner points of $\Delta_f(w_1', w_2', w_3')$ and since (w_1, w_1', w_2', w_2), (w_2, w_2', w_3', w_3) and (w_3, w_3', w_1', w_1) are geodesic paths, wz_1', wz_2', wz_3' are $\frac{\alpha}{2}$-inner points of $\Delta_f(w_1, w_2, w_3)$. It follows that $X * Y$ is L'_α. □

Theorem 8.
A free product of quasi-median spaces is quasi-median for the free product metric.

Proof of Theorem 8. Let $\delta, \delta' \geq 0$ and assume that (X, d_X) is δ-median and (Y, d_Y) is δ'-median. By Theorem 2.5.6, (X, d_X) and (Y, d_Y) are α-median where $\alpha = \max(\delta, \delta')$ and denote by C the **(α-Med2)** constant of X and Y.
As (X, d_X), (Y, d_Y) are L'_α, by Proposition 2.6.19, $(X * Y, d_{X*Y})$ is L'_α. Moreover, we have, for all $x, x', x'' \in X$ and all $y, y', y'' \in Y$:

$$\text{diam}(M_\alpha(x, x', x'')) \leq C\delta \text{ and } \text{diam}(M_\alpha(y, y', y'')) \leq C\delta.$$

Then, by Corollary 2.6.18, for all $w_1, w_2, w_3 \in X * Y$,

$$\text{diam}(M_\alpha(w_1, w_2, w_3)) \leq (C + 1)\delta.$$

As a conclusion, $(X * Y, d_{X*Y})$ is α-median with **(α-Med2)** equal to $C + 1$. □

Corollary 2.6.20. *If $(X, d_X), (Y, d_Y)$ have bounded diameter, then $X * Y$ is a δ-median space for $\delta = \max(diam(X), diam(Y))$.*

Proof. If (Z, d_Z) has bounded diameter, then Z is α-median for $\alpha = \text{diam}(Z)$. Hence, the statement follows immediately from Theorem 8. □

Chapter 3

A plig metric on compactly generated groups

3.1 Introduction

The goal of this chapter is to give an explicit construction of a proper left-invariant metric which generates the topology on locally compact, compactly generated groups which only depends on the Haar measure on the group and on the notion of V-path for a well chosen generating set V.

Let G be a compactly generated group and consider V a compact symmetric generating neighbourhood of the identity element e of G. Since V generates the group, every element of G can be written as a word in V. To this word, we associate a path from the identity to the element x called V-path from e to x, namely, a finite sequence of elements of G, $(w_0, ..., w_n)$ such that $w_0 = e$, $w_n = x$, and $w_{i-1}^{-1} w_i \in V$ for $i = 1, ..., n$. More precisely, given a word $x = v_1 ... v_n$ where $v_i \in V$, the V-path associated with this word is $w_0 = e$ and $w_i = w_{i-1} v_i$ for $i = 1, ..., n$. Conversely, a V-path $(w_0, ..., w_n)$ from e to x gives rise to a word representing x in the generating set V : $x = v_1 ... v_n$ where $v_i = w_{i-1}^{-1} w_i$.

The equivalent notion of word in V/V-path allows us to define a length function l on G :

$$l_V(x) = \inf\{n \in \mathbb{N} \mid x \in V^n\} = \inf\{n \in \mathbb{N} \mid \exists (w_0, ..., w_n) \; V\text{-path from e to } x\}.$$

The metric $d(x, y) = l(x^{-1}y)$ on G induced by this length function is left-invariant and proper but, clearly, in the non discrete case, it does not generate the topology of the group. This metric, called the word metric on G, only depends on the notion of words in V. In the same philosophy, we want to define, on compactly generated groups, a "canonical" metric which only depends on the notion of path on a compact generating set and which has properties as good as the word metric in the finitely generated case : properness, left-invariance and generation of the topology.

Subsequently, we assume that G is a locally compact, compactly generated group.

For simplicity, we say that V is a ***csg*** neighbourhood when V is a *compact symmetric generating* neighbourhood in G.

3.2 Preliminaries

On a locally compact group G, there exists a unique (up to a multiplicative constant) left-invariant measure called the Haar measure. If V is a csg neighbourhood of e, we consider μ_V the left-invariant Haar measure on G such that $\mu_V(V) = 1$.

Subsequently, we will use the convenient terminology of [HP06] : we say that a metric on G is a **plig** metric if it is a *proper left-invariant metric that generates the topology of G.*

Definition 3.2.1. (V-paths/words in V)
Let V be a csg neighbourhood of e.

- *Let x, y be elements of G. A finite sequence $(w_0, ..., w_n)$ of elements of G where $n \in \mathbb{N}$ is called a* V-path *of length n from x to y if $w_0 = x$, $w_n = y$ and $w_{i-1}^{-1}w_i \in V \setminus \{e\}$ for $i = 1, ..., n$.*
- *Let x be an element of G, $v_1, ..., v_n$ be elements of $V \setminus \{e\}$ where $n \in \mathbb{N}$. We say that x is represented by the* word $v_1...v_n$ *of length n if $x = v_1...v_n$.*

Remark 3.2.2.
- As said in the introduction, to every word in V of length n representing $x \in G$, we can associate a V-path of length n from e to x, and conversely : if $x = v_1...v_n$ is a word in V, we get a V-path from e to x by setting $w_0 = e$ and, for $i = 1, ..., n$, $w_i = v_1...v_i$; and if $(w_0, ..., w_n)$ is a V-path from e to x, we get a word $x = v_1...v_n$ by setting $v_i = w_{i-1}^{-1}w_i$.

- Since V generates the group, for every $x \in G$, there exists $n \in \mathbb{N}$ such that x is represented by a word in V of length n, and then, by the previous remark, there exists a V-path of length n from e to x.

Definition-Proposition 3.2.3. (word length/word metric)
Let x, y be elements of G. We call word length *of x the quantity $l_V(x) = \inf\{n \in \mathbb{N} \mid x \in V^n\}$ and* word metric *the function δ_V on $G \times G$ such that $\delta_V(x, y) = l(x^{-1}y)$.*
Then l_V is a length function on G, δ_V is a proper left-invariant metric on G and we have :

$$l_V(x) = \inf\{n \in \mathbb{N} \mid x = v_1...v_n \textit{ word in } V\} = \inf\{n \in \mathbb{N} \mid (w_0, ..., w_n)\ V\textit{-path from } e \textit{ to } x\}.$$

The following lemma is a folkloric result :

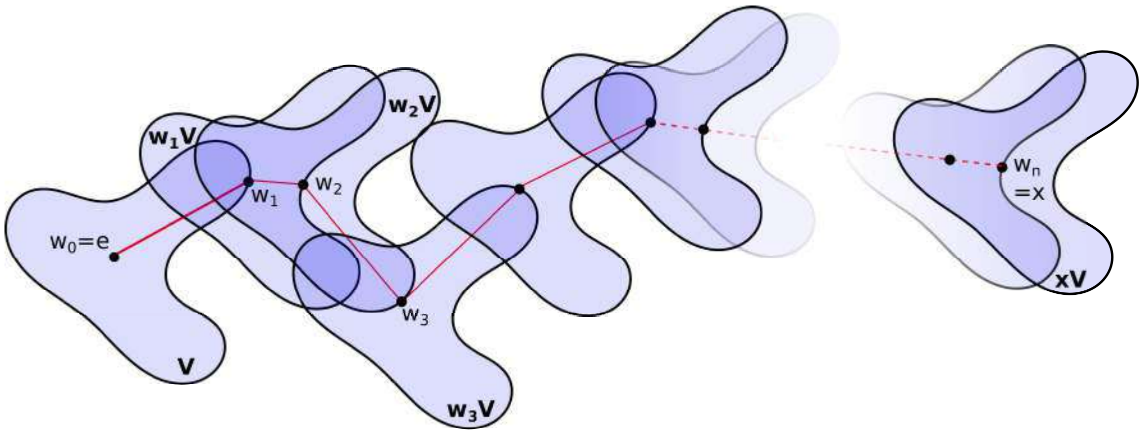
Lemma 3.2.4. *Let V and V' be csg neighbourhoods of e. Then l_V and $l_{V'}$ are bi-Lipschitz.*

Proof. Denote $N = \min\{n \in \mathbb{N} \mid V' \subset V^n\}$. Since V' is compact and V generates G, we have $N < +\infty$: indeed, $\{x\overset{\circ}{V}\}_{x \in V'}$ is an open cover of V', then there exists $x_1, ..., x_k \in V'$ such that $V' \subset \bigcup_{i=0}^{k} x_i V$. Set $N_0 = \max\{n \in \mathbb{N} \mid x_i \in V^n,\ i = 1, ..., k\}$ which is clearly finite since V generates G. We have $N \leq N_0 + 1$ since every element of V' can be written as $x_i y$ for some $y \in V$. Moreover, remark that $N = \max\{l_V(x) \mid x \in V'\}$.
Let $x \in G$ and $x = x'_1...x'_n$ be a word in V' of minimal length (i.e. $l_{V'}(x) = n$). We can write every x'_i as a word in V of minimal length n_i where $n_i \leq N$ for $i = 1, ..., n$. Hence, $l_V(x) \leq Nn = Nl_{V'}(x)$. By interverting the roles of V and V', we also have $l_{V'}(x) \leq N' l_V(x)$ where $N' = \min\{n \in \mathbb{N} \mid V \subset V'^n\}$. It follows that l_V and $l_{V'}$ are bi-lipschitz. ☐

3.3 A pseudo-metric from V-paths

3.3.1 Definitions

In this section, we give the definition of a pseudo-metric depending on the notion of V-paths between any two points of G.
For $x, y \in G$, consider a V-path $(w_0, ..., w_n)$ between x and y. The union of the left-translates $w_i V$ is a compact set of μ_V-measure bigger than 1 and less than $n + 1$.



Union of the left translates w_iV for a V-path from e to $x \in G$.

This allows us to consider the following quantities :

Definition 3.3.1. *Let V be a csg neighbourhood of e. Let $x, y \in G$ and $(w_0, ..., w_n)$ be a V-path from x to y. We call V-*length *of $(w_0, ..., w_n)$ the quantity :*

$$L_V(w_0, ..., w_n) = \mu_V\left(\bigcup_{i=0}^{n} w_iV\right) - 1,$$

and we set :

$$\rho_V(x, y) = \inf \{L_V(w_0, ..., w_n) \mid (w_0, ..., w_n)\, V\text{-path from } x \text{ to } y\}.$$

Proposition 3.3.2. *Let V be a csg neighbourhood of e. Then ρ_V is a left-invariant pseudo-metric on G.*

Proof. Let $x, y, z \in G$.

- *left-invariance* : As μ_V is left-invariant, L_V is left-invariant. Hence, ρ_V is left-invariant.

- ρ_V *is symmetric* : Let $w = (w_0, ..., w_n)$ be a V-path from x to y and set $w'_i = w_{n-i}$ for $i = 0, ..., n$.
Then $w'_0 = w_n = y$, ${w'}_{i-1}^{-1} w'_i = w_{n-i+1}^{-1} w_{n-i} \in V$ for $i = 1, ...n$ since V is symmetric, and

$w'_n = w_0 = x$. Hence, $(w'_0, ..., w'_n)$ is a V-path from y to x.
In addition, we clearly have $L_V(w'_0, ..., w'_n) = L_V(w_0, ..., w_n)$. It follows that $\rho_V(x,y) = \rho_V(y,x)$.

- *Triangle inequality* : Let $\varepsilon > 0$. Consider V-paths $t = (t_0, ..., t_n)$ from x to y and $s = (s_0, ..., s_m)$ from y to z such that :

$$L_V(t) \leq \rho_V(x,y) + \frac{\varepsilon}{2},$$

$$L_V(s) \leq \rho_V(y,z) + \frac{\varepsilon}{2}.$$

The sequence $w = (w_0, ..., w_{n+m})$ such that $w_i = t_i$ for $i = 0, ..., n$ and $w_{n+i} = s_i$ for $i = 1, ..., m$ is a V-path from x to z.
Moreover, we have $\bigcup\limits_{i=0}^{n+m} w_i V = (\bigcup\limits_{i=0}^{n} t_i V) \cup \big((\bigcup\limits_{i=1}^{m} s_i V) \setminus V\big)$ and then :

$$L_V(w) \leq \mu_V(\bigcup_{i=0}^{n} t_i V) - 1 + \mu_V((\bigcup_{i=1}^{m} s_i V) \setminus V) = \mu_V(\bigcup_{i=0}^{n} t_i V) - 1 + \mu_V(\bigcup_{i=0}^{m} s_i V) - 1.$$

Hence, $L_V(w) \leq \rho_V(x,y) + \rho_V(y,z) + \varepsilon$.
It follows that $\rho_V(x,z) \leq \rho_V(x,y) + \rho_V(y,z)$. □

3.3.2 Properties of the pseudo-metric ρ_V

Subsequently, for a csg neighbourhood V of e, we denote $B_V(x,r) = \{y \mid \rho_V(x,y) \leq r\}$ the ρ_V-ball of center $x \in G$ and radius $r \geq 0$.

Lemma 3.3.3. *Let K be a compact subset of G and μ be a left-invariant Haar measure on G. The following functions from the topological group G to $\mathbb{R}_+$ are continuous :*
— $\phi_K : x \mapsto \mu(K \cap xK)$;
— $\varphi_K : x \mapsto \mu(K \cup xK)$; *and*
— $\psi_K : x \mapsto \mu(K \triangle xK)$.

Proof. Notice that, for every $x \in G$, $\varphi_K(x) = 2\mu(K) - \phi_K(x)$ and $\psi_K(x) = \varphi(x) - \phi_K(x)$. Hence, it is sufficient to show that ϕ_K is continuous.

Let $\varepsilon > 0$. Since μ is outer regular, there exists an open set U such that $K \subset U$ and $\mu(U \setminus K) < \frac{\varepsilon}{2}$.

Moreover, there exists a symmetric neighbourhood W of e (which depends on ε) such that $WV \subset U$ (see [Hal50] Chapter XII, §61).

Let $x, y \in G$ such that $x^{-1}y \in W$. Without loss of generality, we can assume that $\phi_K(x) \geq \phi_K(y)$. Then we have :

$$\begin{aligned}
|\phi_K(x) - \phi_K(y)| &= \mu(K \cap xK) - \mu(K \cap yK), \\
&\leq \mu((K \cap xK) \setminus (K \cap yK)), \\
&\leq \mu((K \cap xK) \triangle (K \cap yK)),
\end{aligned}$$

but $(K \cap xK) \triangle (K \cap yK) = K \cap (xK \triangle yK)$, then :

$$\begin{aligned}
|\phi_K(x) - \phi_K(y)| &\leq \mu(xK \triangle yK), \\
&= \mu(y^{-1}xK \setminus K) + \mu(x^{-1}yK \setminus K) \leq 2\mu(WK \setminus K) \\
|\phi_K(x) - \phi_K(y)| &\leq \mu(U \setminus K) < \varepsilon
\end{aligned}$$

□

Proposition 3.3.4. *Let V be a csg neighbourhood of e. Then the topology induced by ρ_V is coarser than the topology of G.*

Proof. Denote by $\mathcal{T}_G$ the topology of G and let $\mathcal{V}_G(e) = \{U \in \mathcal{T}_G \mid e \in U\}$; then $\mathcal{V}_G(e)$ is a fundamental system of neighbourhoods of e for $\mathcal{T}_G$. Since ρ_V is a pseudo-metric, the family $\mathcal{V}_{\rho_V}(e) = \{B_V(e, r) \mid r > 0\}$ is a fundamental system of neighbourhoods of e for the topology $\mathcal{T}_{\rho_V}$ of G induced by ρ_V.

Let us show that every element of $\mathcal{V}_{\rho_V}(e)$ contains an element of $\mathcal{V}_G(e)$.

Consider the function $f_V : x \mapsto \mu_V(V \cup xV) - 1$ from G to $\mathbb{R}_+$; by Lemma 3.3.3, f_V is continuous with respect to $\mathcal{T}_G$.

Let $r > 0$. Since f_V is continuous, $f_V^{-1}([0, r[)$ belongs to $\mathcal{V}_G(e)$ and we set $U = f_V^{-1}([0, r[) \cap \overset{\circ}{V} \in \mathcal{V}_G(e)$. Then $U \subset B_V(e, r)$: indeed, for all $v \in V$,

$$\rho_V(e, v) = \mu_V(V \cup vV) - 1 = f_V(v),$$

and hence, for all $u \in U$, $\rho_V(e, u) = f_V(u) < r$.

As a consequence, every ball of radius $r > 0$ of $\mathcal{V}_{\rho_V}(e)$ contains an element of $\mathcal{V}_G(e)$.

It follows that $\mathcal{T}_{\rho_V}$ is coarser than $\mathcal{T}_G$. □

Notation 3.3.5. *Let V be a csg neighbourhood of e. For $x \in G$ and $w = (w_0, ..., w_k)$ a V-path from e to x, we denote, for $i \in \mathbb{N}$:*

$$M_i(w) = \{m \in [\![0, k]\!] \mid l_V(w_m) = i\}.$$

Lemma 3.3.6. *Let V be a csg neighbourhood of e. Let $x \in G$ such that $l_V(x) = n$ and $w = (w_0, ..., w_k)$ be a V-path from e to x. Then, for $i = 0, ..., n$,*

$$M_i(w) \neq \emptyset.$$

Proof. The case $x = e$ is trivial and if $x \neq e$, notice that for $i = 0, 1$ and $i = n$, clearly $M_i \neq \emptyset$ since $w_0 = e \in M_0$, $w_1 \in M_1$ and $w_n = x \in M_n$.

Then we have to show this lemma for $n \geq 3$ and $i \in \{2, ..., n-1\}$. Let $n \geq 3$ and assume by contradiction that there exists $j \in \{2, ..., n-1\}$ such that $M_j = \emptyset$. Denote $j_0 = \min\{i \in [\![2, ..., n-1]\!] \mid M_i = \emptyset\} \leq j$.

By definition of j_0, $M_{j_0-1} \neq \emptyset$ and consider $m = \max M_{j_0-1} \geq 1$. Then we have $l_V(w_m) = j_0 - 1$ and there exists $v \in V$ such that $w_{m+1} = w_m v \in V^{j_0}$. Then, as $M_{j_0} = \emptyset$ and m is maximal, we have, for every $1 \leq l \leq k - m$:

$$l_V(w_{m+l}) < j_0 - 1.$$

It follows that $n = l_V(w_k) < j_0 - 1$ which is a contradiction. □

Notation 3.3.7. *Let V be a csg neighbourhood of e. Let $x \in G$ such that $l_V(x) = n$ and $w = (w_0, ..., w_k)$ be a V-path from e to x. We denote, for $i = 0, ..., n$:*

$$m_i(w) = \max M_i(w) = \max\{m \in [\![0, k]\!] \mid l_V(w_m) = i\}.$$

Remark 3.3.8. *we have $0 \leq m_0(w) < m_1(w) < ... < m_n(w) = k$.*

Lemma 3.3.9. *Let V be a csg neighbourhood of e. Let $n \in \mathbb{N}$ and $x \in G$ such that $l_V(x) = 3n$. Then $\rho_V(e, x) \geq n$.*

Proof. Let $x \in G$ such that $l_V(x) = 3n$ and $w = (w_0, ..., w_k)$ be a V-path from e to x. Denote $t_i = w_{m_{3i}(w)}$ for $i = 0, ..., n$. Let us show that, for $i = 0, ..., n-1$, $t_i V \cap t_{i+1} V = \emptyset$.

Assume by contradiction that there exists $v, v' \in V$ such that $t_{i+1} v = t_i v'$. Then, as $l_V(t_i) = 3i$, $t_{i+1} = t_i v' v^{-1}$ belongs to V^{3i+2} which is a contradiction since $l_V(t_{i+1}) = 3i+3$.

Hence $t_iV \cap t_{i+1}V = \emptyset$.

It follows that :

$$\mu_V(\bigcup_{i=0}^{k} w_iV) \geq \mu_V(\bigcup_{i=0}^{n} t_iV) \geq n+1.$$

This is true for all V-path $(w_0, ..., w_k)$ from e to x and then $\rho_V(e,x) \geq n$. $\square$

Proposition 3.3.10. *Let V be a csg neighbourhood of e. Then ρ_V is a* proper *pseudo-metric on G.*

Proof. By Proposition 3.3.4, for all $r \geq 0$, $B_V(e,r)$ is closed for the topology of G. Let us show that, for every $r \geq 0$, there exists a compact set containing $B_V(e,r)$.

Let $r \geq 0$. If $x \in B_V(e,r)$, we have $\rho_V(e,x) < \lfloor r \rfloor + 1$ and then, by Lemma 3.3.9 :

$$l_V(x) < 3(\lfloor r \rfloor + 1).$$

Hence x belongs to $V^{3(\lfloor r \rfloor+1)}$ which is a compact set since V is compact. $\square$

Proposition 3.3.11. *Let V be a csg neighbourhood of e. Then (G,ρ_V) and (G,δ_V) are quasi-isometric.*

Proof. Let $x \in G$ of word length n. Let $(w_0, ..., w_n)$ be a V-path from e to x of length n. We have :

$$\rho_V(e,x) \leq \mu_V(\bigcup_{i=0}^{n} w_iV) - 1 \leq n = l_V(x).$$

Moreover,

$$\rho_V(e,x) \geq \frac{\lfloor n \rfloor}{3} \geq \frac{l_V(x)}{3} - \frac{1}{3}.$$

It follows that $id : G \to G$ is a quasi-isometry between (G,ρ_V) and (G,δ_V). $\square$

Corollary 3.3.12. *Let V, V' be a csg neighbourhoods of the identity. Then (G,ρ_V) and $(G,\rho_{V'})$ are quasi-isometric.*

Proof. By Lemma 3.2.4, l_V and $l_{V'}$ are bi-Lipschitz and then, (G,δ_V) and $(G,\delta_{V'})$ are quasi-isometric. Hence, by transitivity, Proposition 3.3.11 implies that (G,ρ_V) and $(G,\rho_{V'})$ are quasi-isometric. $\square$

From a result of Guivarc'h in [Gui73] (see Theorem 1.1), we deduce that the volume of ρ_V-balls is exponentially controlled :

Theorem 3.3.13 (Guivarc'h)**.** *The sequence* $(\mu_V(V^n)^{\frac{1}{n}})_{n\in\mathbb{N}^*}$ *has a limit larger or equal to* 1*.*

Proposition 3.3.14. *Let* V *be a csg neighbourhood. The* ρ_V*-balls have exponentially controlled growth.*

Proof. Let $l \geq 1$ be the limit of the sequence $(\mu_V(V^n)^{\frac{1}{n}})_{n\in\mathbb{N}^*}$. Then there exists $\alpha > 0$ such that for n big enough,

$$\mu_V(V^n) \leq e^{\alpha n}.$$

By Lemma 3.3.9, for $r > 0$, $B_V(e,r) \subset V^{3(\lfloor r\rfloor+1)}$. It follows that :

$$\mu_V(B_V(e,r)) \leq e^{3\alpha(r+1)}.$$

□

3.4 A plig metric

Without any additional conditions on V, ρ_V is not a metric in general : for instance, if there exists $v \in V \setminus \{e\}$ such that $vV = V$, we can remark that $\rho_V(e,v) = \mu_V(V) - 1 = 0$ and then ρ_V does not separate v from e.

Lemma 3.4.1. *Let* V *be a csg neighbourhood of* e *and* $x \in G$*. Then* $\rho_V(x,e) = 0$ *if, and only if,* $\mu_V(V \bigtriangleup xV) = 0$*.*

Proof. Let $y \in G \setminus V$. Since the topological group G is regular, there exists an open set $U \subset yV$ such that $y \in U$ and $U \cap V = \emptyset$. Then :

$$\mu_V(yV \setminus V) \geq \mu_V(U) > 0.$$

Hence we have $\rho_V(y,e) \geq \mu_V(yV \setminus V) > 0$ and $\mu_V(V \bigtriangleup xV) \geq \mu_V(yV \setminus V) > 0$. As a consequence, we have to prove our statement for $x \in V$.
For all $x \in V$, $\rho_V(e,x) = \mu_V(V \cup xV) - 1 = \mu_V(xV \setminus V)$ and, by left-invariance of μ_V, we also have $\rho_V(e,x) = \mu_V(V \setminus xV)$.
It follows that :

$$\rho_V(e,x) = \frac{1}{2}\mu_V(V \bigtriangleup xV).$$

As a consequence, $\rho_V(x,e) = 0$ if, and only if, $\mu_V(V \bigtriangleup xV) = 0$. □

Definition 3.4.2. *We say that a compact neighbourhood V of e is* mobile *if, for every $x \in V \setminus \{e\}$,*

$$\mu_V(V \triangle xV) > 0.$$

Remark 3.4.3.
(1) In $G = \mathbb{C}^*$, the csg neighbourhood $V = \{z \in \mathbb{C} \mid \frac{1}{2} \leq |z| \leq 2\}$ is not mobile : indeed, for $v = e^{i\theta} \in V$, $vV = V$.

(2) It is clear that if V is mobile, $V \subsetneq G$. Then, subsequently, we exclude the cases where $\#G = 1, 2$ or 3.

Lemma 3.4.4. *Let V be a csg neighbourhood of e. The set $K_V = \{x \in V \mid \mu_V(V \triangle xV) = 0\}$ is a compact subgroup of G.*

Proof. The set K_V is a subgroup of G as the stabilizer for the regular representation of G on $L^1(G, \mu_V)$ of $\mathbb{1}_V$. Moreover, K_V is contained in the compact set V and it is a closed set as the reverse image of $\{0\}$ by the continuous function $\psi_K : x \mapsto \mu(V \triangle xV)$ (see Lemma 3.3.3). Hence, K_V is compact.

□

Proposition 3.4.5. *Let V be a csg neighbourhood of e.*

Then ρ_V is a left-invariant metric *on G if, and only if, V is mobile.*

Proof. By Proposition 3.3.2, ρ_V is a left-invariant pseudo-metric. Moreover, it follows by Lemma 3.4.1 that ρ_V separates points of G if, and only if V is mobile. □

Proposition 3.4.6. *Let V be a csg* mobile *neighbourhood of e. Then ρ_V generates the topology of G.*

Proof. We keep the notation of the proof of Proposition 3.3.4. By Proposition 3.3.4, the topology $\mathcal{T}_{\rho_V}$ induced by ρ_V is coarser than the topology $\mathcal{T}_G$ of G. It remains to prove that $\mathcal{T}_G$ is coarser than $\mathcal{T}_{\rho_V}$.

Let us consider the $\mathcal{T}_G$-continuous function $f_V : x \mapsto \mu_V(V \cup xV) - 1$. Notice that if $x \notin V^2$, then $f_V(x) = 1$. In fact, if $x \notin V^2$, $V \cap xV = \emptyset$.

Let $U \in \mathcal{V}_G(e)$. Let us show that there exists $R > 0$ such that $f_V^{-1}([0, R[) \subset U$. Let $(x_n)_{n \in \mathbb{N}^*}$ be a G valued sequence such that, for $n \in \mathbb{N}^*$, $f_V(x_n) < \frac{1}{n}$. By the previous remark, for every n, $x_n \in V^2$.

Assume by contradiction that there exists $N \in \mathbb{N}^*$ such that for all $n \geq N$, $x_n \notin U$. As

V^2 is compact and U^c is closed, there exists a subsequence of (x_n) which converges to $x \in U^c$. By continuity of f_V, $f_V(x) = 0$. Moreover, we have :

$$0 = f_V(x) = \mu_V(V \cup xV) - 1 = \frac{1}{2}\mu_V(V \triangle xV).$$

Hence, as V is mobile, $x = e$. Contradiction since $x \notin U$.

It follows that there exists $0 < R < 1$ such that $f_V^{-1}([0, R[) \subset U$. Thus, for every $x \in B_V(e, r)$ with $r < R$, we have $f_V(x) \leq \rho_V(e, x) \leq r < R$ and then $B_V(e, r) \subset f_V^{-1}([0, R[) \subset U$.

Hence, for every U in $\mathcal{V}_G(e)$, there exists $r > 0$ such that the ball $B_V(e, r) \in \mathcal{V}_{\rho_V}(e)$ is contained in U.

It follows that $\mathcal{T}_G$ and $\mathcal{T}_{\rho_V}$ are equivalent and then, ρ_V generates the topology $\mathcal{T}_G$ of G. □

We can now summarise all these statements in the following theorem :

Theorem 9.
Let G be a locally compact, compactly generated group and V be a compact, symmetric, mobile and generating neighbourhood of the identity. Then ρ_V is a **plig** *metric on G for which the balls have exponentially controlled growth.*

Moreover, if V' be a compact, symmetric, mobile and generating neighbourhood of the identity, then (G, ρ_V) and $(G, \rho_{V'})$ are **quasi-isometric**.

Remark 3.4.7. *By a result of Haagerup and Przybyszewska in [HP06] (see Theorem 6.5), given a plig metric on a locally compact G, one can build a proper isometric affine action on the Banach space $\bigoplus_{n\in\mathbb{N}^*} L^{2n}(G)$. As we discuss in Part 4.3.1, it could be interesting to improve this result using the explicit plig metric ρ_V.*

3.5 Optimal V-paths

It is natural to ask wether, for any two elements of G, there exists a V-path joining this two elements which realises the distance between them.

Definition 3.5.1. *Let V be a csg neighbourhood of e and $x, y \in G$.*
We say that a V-path $(w_0, ..., w_n)$ from x to y is minimal *if $l_V(x^{-1}y) = n$.*
We say that a V-path $(w_0, ..., w_n)$ from x to y is optimal *if $\rho_V(x, y) = \mu_V(\cup_{i=0}^n w_i V) - 1$.*

We show, in the following lemma, that for a fixed x of G, the distance $\rho_V(e,x)$ can be obtained considering only V-paths from e to x of bounded length (where the bound depends on V and the word length of x). A way to prove this is, given a V-path from e to x of length big enough, to shorten this path while preserving its contribution in the distance $\rho_V(e,x)$.

Lemma 3.5.2. *Let V be a csg neighbourhood of e and $x, y \in G$.*
There exists an integer N such that :

$$\rho_V(x,y) = \inf \left\{ \mu_V \left(\bigcup_{i=0}^{n} w_i V \right) \mid (w_0, ..., w_n) \, V\text{-path from } x \text{ to } y,\ n \leq N \right\} - 1$$

.

Proof. Let $x \in G$ such that $l_V(x) = n \in \mathbb{N}^*$. We have $\lfloor \frac{n}{3} \rfloor \leq \rho_V(e,x) \leq n$ and notice that, for a V-path $(w_0, ..., w_k)$ from e to x, if there exists $1 \leq j \leq k$ such that $l_V(w_j) > 3(n+1)$, then :

$$\mu_V(\cup_{i=0}^{k} w_i V) - 1 \geq \mu_V(\cup_{i=0}^{j} w_i V) - 1 \geq \rho_V(e, w_j) \geq n+1 \geq \rho_V(e,x) + 1.$$

It follows that the quantity $\rho_V(e,x)$ can be obtained considering only the infimum among V-paths $(w_0, ..., w_k)$ such that for $j = 0, ..., k$, $l_V(w_j) \leq 3(n+1)$.

The set $V^{3(n+1)}$ is compact and, by continuity of $(x,y) \mapsto x^{-1}y$, there exists a symmetric open neghibourhood U of e such that $U \subset U^2 \subset V$ (see [Die69] Chapter XII, §8). As $\{yU\}_{y \in V^{3(n+1)}}$ is an open cover of $V^{3(n+1)}$, there exists $y_1, ..., y_m \in V^{3(n+1)}$ such that $V^{3(n+1)} \subset \cup_{i=1}^{m} y_i U$.

Let $(w_0, ..., w_k)$ be a V-path from e to x such that for $j = 0, ..., k$, $l_V(w_j) \leq 3(n+1)$. Assume that $k \geq 3m$. Then there exists $i \in [\![1, m]\!]$ such that $\#\{w_j \mid w_j \in y_i U\} \geq 3$. We denote by j_0 the smallest index j such that $w_j \in y_i U$ and by j_1 the biggest of such indices. We have $w_{j_0} = y_i u_0$, $w_{j_1} = y_i u_1$ for some $u_0, u_1 \in U$ and then :

$$w_{j_0}^{-1} w_{j_1} = u_0^{-1} u_1 \in U^2 \subset V.$$

Hence, $w' = (w_0, ..., w_{j_0}, w_{j_1}, ..., w_k) =: (w'_0, ..., w'_{k'})$ is a V-path from e to x which length k' satisfies $k' \leq k-1$ since there is at least one w_j with $j_0 < j < j_1$ and we clearly have :

$$\mu_V(\cup_{i=0}^{k} w_i V) \geq \mu_V(\cup_{i=0}^{j_0} w_i V \cup \cup_{i=j_1}^{k} w_i V) = \mu_V(\cup_{i=0}^{k'} w'_i V).$$

If the length k' of w' is still bigger than $3m$, we shorten w' in the same way, and then, by induction, we obtain a V-path $w^* = (w_0^*, ..., w_{k^*}^*)$ from e to x of length k^* less than $N = 3m$ and such that :

$$\mu_V(\cup_{i=0}^{k} w_i V) \geq \mu_V(\cup_{i=0}^{k^*} w_i^* V).$$

□

Lemma 3.5.3. *Let V be a csg neighbourhood of e and $n \in \mathbb{N}^*$. The function ϕ from $G^{\times n} = \underbrace{G \times ... \times G}_{n\ times}$ to $\mathbb{R}_+$ defined by $\phi(g_1, ..., g_n) = \mu_V(\cup_{i=1}^{n} g_i V) - 1$ is continuous for product topology.*

Proof. Let $\varepsilon > 0$. There exists an open set U such that $V \subset U$ and $\mu_V(U \smallsetminus V) < \frac{\varepsilon}{n}$ by outer regularity of μ_V. Consider a symmetric neighbourhood W of e such that $WV \subset U$. Then $W^{\times n}$ is a neighbourhood of the identity $(e, ..., e)$ of the topological group $G^{\times n}$. Let $g_1, ..., g_n, h_1, ..., h_n \in G$ such that $h_i^{-1} g_i \in W$ for $i = 1, ..., n$.
Without loss of generality, we can assume that $\phi(g_1, ..., g_n) \geq \phi(h_1, ..., h_n)$. Then :

$$\begin{aligned} |\phi(g_1, ..., g_n) - \phi(h_1, ..., h_n)| &= \mu_V(\cup_{i=1}^{n} g_i V) - \mu_V(\cup_{i=1}^{n} h_i V) \\ &\leq \mu_V((\cup_{i=1}^{n} g_i V) \smallsetminus (\cup_{i=1}^{n} h_i V)) \\ &\leq \mu_V(\cup_{i=1}^{n} (g_i V \smallsetminus h_i V)) \\ &\leq \textstyle\sum_{i=1}^{n} \mu_V(g_i V \smallsetminus h_i V) = \sum_{i=1}^{n} \mu_V(h_i^{-1} g_i V \smallsetminus V) \\ &\leq \textstyle\sum_{i=1}^{n} \mu_V(WV \smallsetminus V) \\ &\leq n\, \mu_V(U \smallsetminus K) < \varepsilon \end{aligned}$$

Then ϕ is continuous. □

Theorem 10.
Let V be a csg neighbourhood of e. For every $x \in G$, there exists an optimal V-path from e to x.

Proof of Theorem 10. Let $x \in G$. Let N be the integer given by Lemma 3.5.2 and denote :

$$P_x^N = \{(w_0, ..., w_N) \mid \exists k \leq N,\ (w_0, ..., w_k) \text{ is a } V\text{-path from } e \text{ to } x \text{ and } w_{k+1} = ... = w_N = w_k\},$$

and

$$W_x^N = \{(v_1, ..., v_N) \mid v_i \in V \text{ for } i = 1, ..., N \text{ and } v_1 ... v_N = x\}.$$

Consider the topological groups $G^{\times N}$ and $G^{\times (N+1)}$ respectively endowed with the product topology of G. Then W_x^N is a compact subset of $G^{\times N}$. In fact, $W_x^N \subset V^{\times N}$ which is compact in $G^{\times N}$ as a product of compact sets of G and $W_x^N = F^{-1}(\{x\})$, where F :

$G^{\times N} \to G$ is the continuous function defined by $F(g_1, ..., g_N) = g_1 ... g_N$.
It follows that P_x^N is a compact subset of $G^{\times(N+1)}$ as the direct image of W_x^N by the continuous function $(g_1, ..., g_N) \mapsto (e, g_1, g_1 g_2, ..., g_1 g_2 ... g_N)$.

By Lemma 3.5.2, we have :

$$\rho_V(e, x) = \inf \phi(P_x^N)$$

where $\phi : (g_0, ..., g_N) \mapsto \mu_V(\cup_{i=0}^N g_i V) - 1$ is continuous by Lemma 3.5.3, and hence, since P_x^N is compact, this infimum is reached by an element $(w_0, ..., w_N)$ of P_x^N which gives rise to a V-path $(w_0, ..., w_k)$ from e to x such that :

$$\rho_V(e, x) = \mu_V(\cup_{i=0}^k w_i V) - 1.$$

$\square$

Perspectives

In this part, we list some questions and developements about the notions we consider in this thesis.

4.1 Questions and developements about the structure of space with labelled partitions

4.1.1 Structure of labelled partitions on graphs and manifolds

We give here a way to build a ℓ^q-structure of labelled partitions on a graph. We consider a real valued function on the set of vertices such that its differential satisfies a ℓ^q condition.

Definition 4.1.1. *Let $\Gamma = (V, \mathbb{E})$ be a graph with a fixed orientation on edges and let $f : V \to \mathbb{R}$. For $e = (e^-, e^+) \in \mathbb{E}$, we denote :*

$$df(e) := f(e^-) - f(e^+).$$

The map $df : \mathbb{E} \to \mathbb{R}$ is called the differential *of f on $\mathbb{E}$.*

Let $\Gamma = (V, \mathbb{E})$ be a connected graph and G be a group acting isometrically on this graph. We consider a function $f : V \to \mathbb{R}$ and we set :

$$\mathcal{P} = \{g.f \mid g \in G\},$$

where $g.f(v) = f(g^{-1}v)$, for $v \in V$ and $g \in G$.
Let $H = \mathrm{Stab}(f) < G$ and notice that $\mathcal{P} \simeq G/H$.

Proposition 4.1.2. *Let $q \geq 1$. Let S be any system of representatives of G/H and* *assume :*

$$\sum_{g\in S}|df(g^{-1}e)|^q < +\infty. \qquad (*)$$

Then $(\Gamma,\mathcal{P},\ell^q(\mathcal{P}))$ *is a structure of space with labelled partitions where* $\ell^q(\mathcal{P})\simeq\ell^q(G/H)$ *and* G *acts by automorphisms on* $(\Gamma,\mathcal{P},\ell^q(\mathcal{P}))$*.*

Proof. Let $v,v'\in V$ and consider an edge-path $\mathrm{ep}(v,v')$ connecting v to v'. For $p\in\mathcal{P}$, we have :

$$c(v,v')(p) = p(v)-p(v') = \sum_{e\in\mathrm{ep}(v,v')} dp(e).$$

Hence :

$$\begin{array}{rcl}
\|c(v,v')\|_q^q & = & \sum_{p\in\mathcal{P}}|p(v)-p(v')|^q, \\
 & = & \sum_{p\in\mathcal{P}}|\sum_{e\in\mathrm{ep}(v,v')} dp(e)|^q, \\
 & \leq & 2^p\sum_{e\in\mathrm{ep}(v,v')}\sum_{[g]\in G/H}|d(g.f)(e)|^q, \\
\|c(v,v')\|_q^q & \leq & 2^p\sum_{e\in\mathrm{ep}(v,v')}\sum_{g\in S}|df(g^{-1}e)|^q < +\infty \text{ by } (*).
\end{array}$$

It follows that $(\Gamma,\mathcal{P},\ell^q(\mathcal{P}))$ is a space with labelled partitions and the natural action of G on $\mathcal{P}$ induces an action by automorphisms on it. $\square$

The previous proposition gives a way to produce structures of labelled partitions on graphs and more generally, by considering a similar ℓ^p condition (see condition $(*)$ above) for a differentiable function, on manifolds. On Gromov hyperbolic groups, following results of Bourdon in [Bou11] or Nica in [Nic12], such functions can be determine in terms of the boundary of the group and of its ℓ^p-cohomology.
It would be interesting to find new examples of spaces with labelled partitions from functions with such ℓ^p condition on the differential and given by the geometry of the space. In the case of the complex hyperbolic space, the metric $\sqrt{d_{\mathrm{hyp}}}$, where d_{hyp} is the hyperbolic metric, can be obtained from a structure of space with measure walls. It is not known wether d_{hyp} itself can be realized as wall metric. We can ask a similar question in terms of labelled partitions metric :

Question 4.1.3. *Is there exists a structure of labelled partitions given by the geometry of the complex hyperbolic space* $\mathcal{H}^n_{\mathbb{C}}$ *which gives a proper isometric affine action of* $SU(n,1)$ *on a Hilbert space ?*

4.1.2 Other constructions of spaces with labelled partitions

1. Semi-direct product and extension of groups

In Part 1.4.3 we discuss a construction of a space with labelled partitions for the semi-direct product which "preserves" the properness of each factor's action. A key hypothesis in Theorem 3 is that $G_1 \rtimes_\rho G_2$ acts by automorphisms on the space with labelled partitions associated with G_1. Then we can ask the following :

Question 4.1.4. *What are the conditions on ρ which ensure the existence of a space with labelled partitions on which $G_1 \rtimes_\rho G_2$ acts by automorphisms and such that the G_1-action on it is proper ?*

Such conditions would give an answer to the following question asked by Valette in [CCJ+01] for the Haagerup property (property PL^2) :

Question 4.1.5. *Let G_1, G_2 be groups with property PL^p and $\rho : G_2 \to Aut(G_1)$ be a morphism. What are the conditions on ρ which ensures that $G_1 \rtimes_\rho G_2$ has property PL^p.*

In this context, the Haagerup property is still unknown for braid groups B_n, for $n \geq 4$. The pure braid group P_n is a finite index subgroup of B_n and can be viewed has the semi-direct product $F_{n-1} \rtimes P_{n-1}$.

Question 4.1.6. *Let $n \geq 4$. Does the pure braid group P_n has Haagerup property ?*

To illustrate Question 4.1.4, we refer to Theorem 1.4.12 : assume I is a G_2-set and G_1 acts properly by automorphisms on $(G_1, \mathcal{P}, F(\mathcal{P}))$. Now consider the space with labelled partitions on $\bigoplus_I G_1$ given by Theorem 1.4.12. Then $\bigoplus_I G_1$ acts properly by automorphisms on it but the G_2-action on $\bigoplus_I G_1$ by shift does not induce an action by automorphisms.
In light of the structure of space with walls given in [CSV12] (see Theorem 1.5.3), a first step in answering Question 4.1.4 is the case of wreath product : How can we build a space with labelled partitions stable by the G_2-action by shift and such that G_1 acts properly on it ?
Notice that in the case of the permutational wreath product $H \wr_I G$, Chifan and Ioana in [CI11] give an obstruction to the stability of Haagerup property. If the pair (G, I) has relative property (T), then $H \wr_I G$ does not have the Haagerup property.
What are then the obstructions in the case of the permutational wreath product of groups with property PL^p ?

Futhermore, semi-direct products are particular cases of group extensions. For a group extension $1 \to H \to G \to K \to 1$, we can wonder if there is a result similar to the case the semi-direct product (see Theorem 3) : given structures of spaces with labelled partitons for H and K under which conditions can we build a space with labelled partitions for G from this structures that perserves properness properties ?

2. Amalgamated products and HNN extensions

We can replace the construction of free product of spaces with labelled partitions in Section 1.6 to the notion of “tree of spaces with labelled partitions“ :

Definition 4.1.7. *Let $T = (V, \mathbb{E})$ be a tree with fixed edges orientation, $((X_v, d_v))_{v\in V}$ and $((X_e, d_e))_{e\in\mathbb{E}}$ be collections of non empty sets such that there exists, for all $e = (e^-, e^+) \in \mathbb{E}$:*

$$F_{e^-} : X_e \hookrightarrow X_{e^-} \text{ and } F_{e^+} : X_e \hookrightarrow X_{e^+}.$$

The tree of spaces associated with T and the collections of sets $((X_v, d_v))_{v\in V}$ and $((X_e, d_e))_{e\in\mathbb{E}}$ is the set X defined by :

$$X = \left(\bigsqcup_{v\in V} X_v \sqcup \bigsqcup_{e\in\mathbb{E}} (X_e \times [0,1]) \right) / \sim,$$

where the identification $\sim$ is given by, for $e = (v^-, v^+) \in \mathbb{E}$,

$$X_e \times \{0\} \sim F_{e^-}(X_e) \text{ and } X_e \times \{1\} \sim F_{e^+}(X_e).$$

Consider a tree $T = (V, \mathbb{E})$, a collection of non empty spaces with labelled partitions $(X_v, \mathcal{P}_v, F_v(\mathcal{P}_v))_{v\in V}$ and a colletion of singletons $(\{\bullet\})_{e\in\mathbb{E}}$. For each edge $e = (e^-, e^+)$ of T, we choose two elements $x^0_{e^-} \in X_{e^-}$, $x^0_{e^+} \in X_{e^+}$ and we consider the trivial embeddings :

$$F_{e^-} : \bullet \mapsto x^0_{e^-} \text{ and } F_{e^+} : \bullet \mapsto x^0_{e^+}.$$

As in Part 1.6.2, we can extend a partition of a vertex set X_v to a partition of the tree of spaces X. Hence, the space with labelled partition on each vertex set induces a natural space with labelled partitions on the tree of space X.
Moreover, the canonical structure of space with walls of the tree T induces a structure of space with walls (X, W) on X. Viewed a space with labelled partitions, the structure

(X, W) is the analog of the space with labelled partitions we consider in Definition 1.6.26. When put together, the previous two structures of labelled partitions on the tree of spaces X induce a proper pseudo-metric of labelled partitions on X.

We want to generalize this construction of space with labelled partitions on a tree of spaces when the edge spaces are no longer reduced to a point.

Question 4.1.8. *What are the conditions on the edge embeddings F_{e^-} and F_{e^+} that guarantee the extensions of the labelled partitions on the vertex sets and edge sets to the all tree of spaces ?*

Amalgamated products and HNN-extensions of groups act naturally on a tree of spaces associated with their Bass-Serre tree (see [SW79]). An answer to Question 4.1.8 could provide conditions for the stability of property PL^p by amalgamated product and HNN-extensions in terms of the subgroups considered.
More precisely, in the case of an amalgamated product $A *_C B$, a way to produce compatible labelled partitions on the tree of spaces could be the following :
Assume there exists structures of spaces with labelled partitions on A/C and B/C such that the respective quotient actions of A and B induce proper actions by automorphisms of spaces with labelled partitions. Taking the pullbacks of the structures of space with labelled partitions by the canonical quotient maps $\pi_A : A \twoheadrightarrow A/C$ and $\pi_B : B \twoheadrightarrow B/C$, we can define a structure of labelled partitions compatible with the embeddings $C \hookrightarrow A$ and $C \hookrightarrow B$.

3. Relatively hyperbolic groups

Question 4.1.9. *Does a relatively hyperbolic group with peripheral subgroups having property PL^p admit property PL^p ?*

In [Hum11], Hume proved that a relatively hyperbolic group admits coarse embeddings into ℓ^p-spaces providing its maximal peripheral subgroups do - and moreover, he states that its ℓ^p compression exponent is equal to the minimum of the exponents of its maximal peripheral subgroups.
In the previous question, we asked about the equivariant case. A way to answer this would be to build a space with labelled partitions, given structures of spaces with labelled partitions on the peripheral subgroups $\{H_i\}$, on which the relatively hyperbolic group G act properly by automorphisms. We could use the hyperbolicity of the Cayley graph

Cay$(G, S \cup \{H_i\})$ to perform such a construction. Notice that a difficulty arises from this point of view : the Caley graph mentionned before is not uniformly locally finite. A recent result of Dreesen in [Dre13] could help to overcome this difficulty. He generalized Bourdon's result in [Bou11] in the following way : any locally compact hyperbolic group admits a proper isometric affine action on an L^p space for p larger than the Ahlfors regular conformal dimension of ∂G.

4.2 Questions and developements about δ-median spaces

4.2.1 L_δ spaces

As we said in the introduction of Chapter 2, we can consider a weaker version of Definition 2.3.6 for quasi-median spaces by replacing the L'_δ condition by L_δ. A natural question to ask is the following :

Question 4.2.1. *Is Theorem 7 still valid when replacing L'_δ by L_δ in the quasi-median definition ?*

A key ingredient in the proof of Theorem 2.5.6 is that in L'_δ spaces, a thick interval is included in some neighbourhood of an interval (see Proposition 2.3.5). In L_δ spaces, this fact is no longer true in general.

4.2.2 Convex gluing

In Theorem 8, we considered a gluing of quasi-median spaces by identifying singletons. We want to generalize this theorem in the case of an identification on a quasi-convex subspace for an appropriate notion of quasi-convexity in the settings of quasi-median spaces :

Question 4.2.2. *Let A, B be quasi-median spaces and $C \hookrightarrow A$, $C \hookrightarrow B$ be isometric embeddings. What are the conditions on C which ensure that $(A \cup B)/C$ is quasi-median ?*

4.2.3 Boundary and structure of space with labelled partitions

In light of the works of Bourdon in [Bou11] and Nica in [Nic12], we want to define a L^p structure of labelled partitions on quasi-median spaces. This may be possible with a relevant notion of boundary for quasi-median spaces which would, in one hand, generalize the notion of visual boundary for hyperbolic spaces and, on the other hand, generalize the notion of Roller boundary for median spaces.

Question 4.2.3. *Is there a structure of boundary for quasi median-space ?*

4.3 Developements about plig metrics on compactly generated groups

4.3.1 Action on Banach spaces

As we saw in Part 3.4, Haagerup and Przybyszewska in [HP06] showed that a locally compact group G acts properly by affine isometries on the Banach space $\bigoplus_{n\in\mathbb{N}^*} L^{2n}(G)$ using a plig metric with exponential growth control on the balls. It could be interesting to improve this result using the properties of the pseudo-metric ρ_V for compactly generated groups.

Question 4.3.1. *Under what conditions on the compactly generated group G does there exists a proper action by affine isometries on a finite sum of L^p spaces ?*

4.3.2 Geodesic metric

Question 4.3.2. *What are the conditions on a csg mobile neighbourhood V of e which ensure that there exists geodesic paths for ρ_V ?*

An sufficient condition which guarantees that an optimal V-path $(w_0, ..., w_k)$ is a geodesic path is the following :

$$\text{For } i = 0, ..., k-1,\ w_{i+1}V \cap \bigcup_{j=0}^{i} w_j V \subset w_i V.$$

This condition is not fulfilled in general but this may be true for some V satisfying a "regularity" property :

Let V be a csg neighbourhood of e. We denote $\alpha_V = \sup\{\rho_V(e, v) \mid v \in V\}$ and we consider the following re-scaled pseudo-metric on G :

$$d_V = \frac{1}{\alpha_V}\rho_V.$$

Definition 4.3.3. *We say that a csg neighbourhood V of e is* regular *if*

$$B_{d_V}(e,1) = V.$$

Then We can ask the following :

Question 4.3.4. *Let G be a compactly generated group. Is there exists regular csg neighbourhood of e ?*

A way to produce regular neighbourhood could be to prove that the following sequence actually converges :

Let V be a csg neighbourhood of e, and consider the sequence $(V_n)_{n\in\mathbb{N}}$ of csg neighbourhood of e defined by induction by :

$$\begin{cases} V_0 = V \\ V_{n+1} = B_{d_{V_n}}(e,1) \text{ for } n \geq 0. \end{cases}$$

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Sylvain Arnt

Géométrie à grande échelle et actions isométriques affines sur des espaces de Banach

Dans le premier chapitre, nous définissons la notion d'espaces à partitions pondérées qui généralise la structure d'espaces à murs mesurés et qui fournit un cadre géométrique à l'étude des actions isométriques affines sur des espaces de Banach pour les groupes localement compacts à base dénombrable. Dans un premier temps, nous caractérisons les actions isométriques affines propres sur des espaces de Banach en termes d'actions propres par automorphismes sur des espaces à partitions pondérées. Puis, nous nous intéressons aux structures de partitions pondérées naturelles pour les actions de certaines constructions de groupes : somme directe ; produit semi-directe ; produit en couronne et produit libre. Nous établissons ainsi des résultats de stabilité de la propriété PL^p par ces constructions. Notamment, nous généralisons un résultat de Cornulier, Stalder et Valette de la façon suivante : le produit en couronne d'un groupe ayant la propriété PL^p par un groupe ayant la propriété de Haagerup possède la propriété PL^p.

Dans le deuxième chapitre, nous nous intéressons aux espaces métriques quasi-médians - une généralisation des espaces hyperboliques à la Gromov et des espaces médians - et à leur propriétés. Après l'étude de quelques exemples, nous démontrons qu'un espace δ-médian est δ'-médian pour tout $\delta' \geq \delta$. Ce résultat nous permet par la suite d'établir la stabilité par produit directe et par produit libre d'espaces métriques - notion que nous développons par la même occasion.

Le troisième chapitre est consacré à la définition et l'étude d'une distance propre, invariante à gauche et qui engendre la topologie explicite sur les groupes localement compacts, compactement engendrés. Après avoir montré les propriétés précédentes, nous prouvons que cette distance est quasi-isométrique à la distance des mots sur le groupe et que la croissance du volume des boules est contrôlée exponentiellement.

Large scale geometry and isometric affine actions on Banach spaces

In the first chapter, we define the notion of spaces with labelled partitions which generalizes the structure of spaces with measured walls : it provides a geometric setting to study isometric affine actions on Banach spaces of second countable locally compact groups. First, we characterise isometric affine actions on Banach spaces in terms of proper actions by automorphisms on spaces with labelled partitions. Then, we focus on natural structures of labelled partitions for actions of some group constructions : direct sum ; semi-direct product ; wreath product and free product. We establish stability results for property PL^p by these constructions. Especially, we generalize a result of Cornulier, Stalder and Valette in the following way : the wreath product of a group having property PL^p by a Haagerup group has property PL^p.

In the second chapter, we focus on the notion of quasi-median metric spaces - a generalization of both Gromov hyperbolic spaces and median spaces - and its properties. After the study of some examples, we show that a δ-median space is δ'-median for all $\delta' \geq \delta$. This result gives us a way to establish the stability of the quasi-median property by direct product and by free product of metric spaces - notion that we develop at the same time.

The third chapter is devoted to the definition and the study of an explicit proper, left-invariant metric which generates the topology on locally compact, compactly generated groups. Having showed these properties, we prove that this metric is quasi-isometric to the word metric and that the volume growth of the balls is exponentially controlled.

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