THE HAAGERUP PROPERTY IS NOT INVARIANT UNDER QUASI-ISOMETRY

MATHIEU CARETTE

WITH AN APPENDIX BY SYLVAIN ARNT, THIBAULT PILLON AND ALAIN VALETTE

Abstract. Using the work of Cornulier-Valette and Whyte, we show that neither the Haagerup property nor weak amenability is invariant under quasi-isometry of finitely generated groups.

A central topic in geometric and measured group theory is the study among finitely generated groups of invariants for quasi-isometry (QI) and measure equivalence (ME) respectively. Despite what might seem from the classical setting of lattices in semisimple Lie groups, quasi-isometry does not imply measure equivalence in general nor vice versa. In fact beyond the world of semisimple Lie groups, a comparison of a few basic invariants reveals strikingly little intersection (See [BHV08, CCJJV01, CH14, Fur11] and references therein for the basics on QI and ME invariants).

- QI invariants: amenability, number of ends, growth, hyperbolicity, finite presentability and Dehn function, asymptotic cones,...
- ME invariants: amenability, property (T), ratio of L^2-Betti numbers, the Haagerup property (also called a-(T)-menability), weak amenability and the Cowling-Haagerup constant,...

With the notable exception of amenability, each property listed above as a QI invariant is moreover known not to be a ME invariant. In the other direction, it is known that neither property (T) nor ratio of L^2-Betti numbers are QI invariants. It is therefore natural to ask whether the Haagerup property and weak amenability, which are both natural generalizations of amenability, are also QI invariants.

We show that these two properties are not QI invariants, settling an open problem raised in [CTV07, p. 774].

Theorem 1. *There exists two finitely generated groups Γ, Λ which are quasi-isometric such that Γ has the Haagerup property and is weakly amenable and Λ has neither of those properties.*

Our examples are fundamental groups of graphs of $\mathbf{Z}^2$'s. The quasi-isometric classification of such groups is addressed by Whyte [Why10]. Cornulier-Valette [CV] characterized which such groups have the Haagerup property and which are weakly amenable. I turns out both answers depend on a holonomy map discussed in Section 0.1. We also discuss the key role of locally compact groups in Section 0.2. In particular allowing unimodular compactly generated locally compact groups instead of finitely generated groups, we give examples as in Theorem 1 where both groups are of the form $\mathbf{R}^2 \rtimes F$ where F is a finitely generated free group (see Remark 7).

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In the appendix, Arnt, Pillon and Valette show that for $1 \le p \le 2$ the vanishing of equivariant L^p compression is not a quasi-isometry invariant.

0.1. **Generalized Baumslag-Solitar groups.** Fix $n \in \mathbf{N}$ an integer. We consider the class $\mathcal{GBS}_n$ of groups Γ acting cocompactly on a locally finite tree T such that all (vertex and edge) stabilizers are isomorphic to $\mathbf{Z}^n$. Equivalently, a group Γ is in $\mathcal{GBS}_n$ if it is the fundamental group of a finite graph of groups where all edge and vertex groups are isomorphic to $\mathbf{Z}^n$ [Ser80].

Let Γ and T be as above. Fix a vertex $v \in T$ and an isomorphism $\Gamma_v \cong \mathbf{Z}^n$. The action of Γ on Γ_v by commensuration induces a homomorphism to the abstract commensurator $\mathrm{hol} : \Gamma \to \mathrm{Comm}(\Gamma_v) = \mathrm{GL}_n(\mathbf{Q})$ which we call the **holonomy map** (this is also sometimes called the modular homomorphism, especially for the class $\mathcal{GBS}_1$ [Lev07]). Note that h is well-defined up to conjugation in $\mathrm{GL}_n(\mathbf{Q})$ (with the different possibilities coming from a different choice of basepoint w and a different identification $\Gamma_w \cong \mathbf{Z}^n$). In fact, the commensurability class of Γ_v inside Γ does not depend on the chosen tree T (as soon as Γ is not amenable, equivalently Γ stabilizes no vertex, no line and no end of T) see e.g. the proof of Lemma 8.5 in [GL07]. From now on, we view the holonomy as a map to $\mathrm{GL}_n(\mathbf{R})$.

We say that Γ, Γ' have **Hausdorff equivalent holonomy** if there is a compact subset $K \subset \mathrm{GL}_n(\mathbf{R})$ and some $g \in \mathrm{GL}_n(\mathbf{R})$ such that $\mathrm{hol}(\Gamma) \subset g\,\mathrm{hol}'(\Gamma')g^{-1}K$ and $g\,\mathrm{hol}'(\Gamma')g^{-1} \subset \mathrm{hol}(\Gamma)K$. In other words $\mathrm{hol}(\Gamma)$ is at finite Hausdorff distance from some conjugate $\mathrm{hol}'(\Gamma')$ for the word metric corresponding to some (equivalently any) compact generating set of $\mathrm{GL}_n(\mathbf{R})$.

Whyte showed that the QI classification of $\mathcal{GBS}_n$ groups is essentially governed by the holonomy map.

Theorem 2 ([Why10, Theorem 0.1])**.** *Among the class of groups in $\mathcal{GBS}_n$ whose Bass-Serre tree T has infinitely many ends the following holds:*

(1) *If two groups are quasi-isometric then they have Hausdorff equivalent holonomy.*
(2) *Groups within a given Hausdorff equivalence class of holonomy divide into three quasi-isometry invariant subclasses:*
    (a) *Those which are of the form $\mathbf{Z}^n \rtimes F$ for F a free subgroup of* $\mathrm{GL}_n(\mathbf{Z})$.
    (b) *Those which are virtually ascending HNN-extensions of some endomorphism $E : \mathbf{Z}^n \to \mathbf{Z}^n$. These are classified up to QI in* [FM00].
    (c) *All groups not of the first two forms, all of which are in a single quasi-isometry class.*

Remark 3. Groups of the form (2b) are exactly those groups which are amenable. Groups of the form (2a) have a holonomy with discrete image in $\mathrm{GL}_n(\mathbf{R})$.

Cornulier and Valette showed that both the Haagerup property and weak amenability of a $\mathcal{GBS}_n$ group is determined by the image of its holonomy.

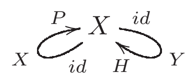
Theorem 4 ([CV, Theorem 1.6])**.** *Let $\Gamma \in \mathcal{GBS}_n$, with holonomy* $\mathrm{hol} : \Gamma \to \mathrm{GL}_n(\mathbf{R})$. *Then the following are equivalent:*

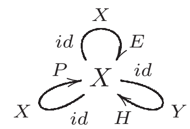
(1) Γ *has the Haagerup property.*
(2) Γ *is weakly amenable.*
(3) Γ *has Cowling-Haagerup constant* 1.
(4) $\overline{\mathrm{hol}(\Gamma)}$ *is amenable.*

Proof of Theorem 1. Let $X = \mathbf{Z}^2 = \langle a, b \mid [a, b]\rangle$ and consider the subgroup $Y = \langle a, b^2\rangle < X$. Consider furthermore the following matrices in $\mathrm{SL}_2(\mathbf{Q})$:

$$H = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}; P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Finally, consider the following graphs of groups

$\mathbf{A} =$ 

and $\mathbf{B} =$ 

and let $\Gamma = \pi_1(\mathbf{A})$ and $\Lambda = \pi_1(\mathbf{B})$. In other words, the groups Γ and Λ are defined by the following presentations:

$$\Gamma = \langle a, b, h, p \mid ab = ba, a^h = a^2, (b^2)^h = b, a^p = a, b^p = ab \rangle$$

$$\Lambda = \langle a, b, h, p, e \mid ab = ba, a^h = a^2, (b^2)^h = b, a^p = a, b^p = ab, a^e = b^{-1}, b^e = a \rangle$$

The holonomy maps of Γ and Λ send $a, b \mapsto 1$ and send h, p, e to the matrices $H, P, E \in \mathrm{SL}_2(\mathbf{R})$ respectively. It follows from the computation

$$H^k P H^{-k} = \begin{pmatrix} 1 & 4^k \\ 0 & 1 \end{pmatrix}$$

that

$$\overline{\mathrm{hol}(\Gamma)} = \overline{\langle H, P \rangle} = \left\{ \begin{pmatrix} 2^k & x \\ 0 & 2^{-k} \end{pmatrix} \middle| \begin{array}{c} k \in \mathbf{Z} \\ x \in \mathbf{R} \end{array} \right\} \tag{1}$$

is solvable and in particular amenable. On the other hand $\langle P, E \rangle = \mathrm{SL}_2(\mathbf{Z})$ contains a free discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$ so that $\overline{\mathrm{hol}(\Lambda)} = \overline{\langle H, P, E \rangle} \subset \mathrm{SL}_2(\mathbf{R})$ is not amenable. In fact $\overline{\langle H, P, E \rangle} = \mathrm{SL}_2(\mathbf{R})$. In view of Theorem 4 the group Γ has the Haagerup property and is weakly amenable, while Λ has neither of these properties.

We now check that Γ and Λ are quasi-isometric using Theorem 2. First observe that the Bass-Serre trees of the graphs of groups $\mathbf{A}$ and $\mathbf{B}$ are the 6 and 8-regular trees respectively, so that they have infinitely many ends. Next observe that there is a compact subset $K \subset \mathrm{SL}_2(\mathbf{R})$ such that $\mathrm{hol}(\Gamma)K = \mathrm{SL}_2(\mathbf{R})$, so in particular Γ and Λ have Hausdorff equivalent holonomies. Finally, $\langle h, p \rangle$ generate free subgroups of Γ and Λ so that neither Γ nor Λ is amenable, and moreover the image of $\langle h, p \rangle$ by the respective holonomies is non-discrete in $\mathrm{GL}_2(\mathbf{R})$. It follows from Remark 3 that both Γ and Λ fall into the class (2c) of Theorem 2 and hence are quasi-isometric to each other. □

Remark 5. While we show that the Cowling-Haagerup constant is not a QI invariant, it follows from Theorem 4 that the class $\mathcal{GBS}_n$ does not contain examples of two quasi-isometric groups with different *finite* Cowling-Haagerup constant.

0.2. **The role of locally compact groups.** For the convenience of the reader, we briefly outline ideas for Cornulier-Valette's equivalence (1) ⇔ (4) in Theorem 4 and Whyte's Theorem 2 (2c) in the special case of our examples with an emphasis towards the role played by non-discrete locally compact groups.

Remark 6. The QI relation naturally extends to compactly generated locally compact (c.g. l.c.) groups, while ME makes sense for unimodular second countable locally compact (u.s.c. l.c.) groups. All invariants mentioned in the introduction extend naturally to these larger settings, with the caveat that amenability is a QI invariant only within *unimodular* c.g. l.c. groups (see [KPV13] for the ME invariance of ratio of L^2-Betti numbers).

Recall the definitions of the groups Γ, Λ and the matrices $H, P, E \in \mathrm{SL}_2(\mathbf{R})$ from the previous section. Let $\varphi : F_2 \to \mathrm{SL}_2(\mathbf{R})$ and $\psi : F_3 \to \mathrm{SL}_2(\mathbf{R})$ be homomorphisms mapping a basis of F_2 to (H, P), respectively a basis of F_3 to (H, P, E). Finally, let $G = \mathbf{R}_2 \rtimes_\varphi F_2$ and $L = \mathbf{R}_2 \rtimes_\psi F_3$.

To show that Γ has the Haagerup property Cornulier and Valette realize Γ as a discrete subgroup in the locally compact group $\mathrm{Aut}(T) \times G$ where T is the (6-regular) Bass-Serre tree of the graph of groups $\mathbf{A}$. They then proceed to show that G has the Haagerup property, and since so does $\mathrm{Aut}(T)$, then Γ has the Haagerup property. On the other hand, the group Λ contains a finite index subgroup of $\mathbf{Z}^2 \rtimes \mathrm{SL}_2(\mathbf{Z})$ which has property (T) relative to the subgroup $\mathbf{Z}^2$, so that Λ cannot have the Haagerup property.

Concerning quasi-isometry, we note that in the forthcoming paper [CT] the author and Tessera use successive cocompact embeddings in (not necessarily unimodular) locally compact groups to show that Γ is quasi-isometric to G and Λ is quasi-isometric to L. Thus, our use of Whyte's result may be reduced to providing a quasi-isometry between the two semidirect products G and L. To do this, Whyte [Why10] produces a carefully chosen bilipschitz bijection $\eta : F_2 \to F_3$ such that there is a compact $K \subset \mathrm{SL}_2(R)$ with the property that $\psi(g)\varphi(\eta(g))^{-1} \in K$ for each $g \in F_2$. This enables him to lift η to a fiber-preserving quasi-isometry $\tilde{\eta} : G \to L$. To produce the bijection, essential use is made of the fact that the maps ψ and φ are not proper.

Remark 7. The semidirect products G and L are arguably simpler (non-discrete) examples showing that the Haagerup property and weak amenability are not QI invariants among *unimodular* s.c. c.g. l.c. groups. Such examples cannot be obtained by cocompact inclusions $H < H'$ as this would force H and H' to be ME. Although it is an example of two quasi-isometric group only one of which is Haagerup, the related cocompact inclusion $\mathbf{R}^2 \rtimes T_2 < \mathbf{R}^2 \rtimes \mathrm{SL}_2(\mathbf{R})$ is uninteresting from the point of view of ME as $\mathbf{R}^2 \rtimes T_2$ is not unimodular (where $T_2 = \overline{\langle H, P \rangle}$ denotes the group described in equation (1)).

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Appendix A. Equivariant compression is not a QI-invariant

by S. Arnt, T. Pillon and A. Valette

Equivariant compression was introduced as a way to quantify the Haagerup property. For G a finitely generated group, $p \in [1, +\infty[$, and $f : G \to L^p$ a G-equivariant map (with respect to some affine isometric action of G on L^p), the L^p-compression of f is:

$$\mathrm{comp}_p(f) = \sup\{\alpha \in [0,1] : \exists C > 0 : C|x^{-1}y|_S^{\alpha} - C \leq \|f(x) - f(y)\|_p \ \forall x, y \in G\},$$

where $|.|_S$ denotes word length with respect to the finite generating set S in G; clearly $\mathrm{comp}_p(f)$ does not depend on the choice of S. Then the L^p-compression of G is $\alpha_p^\sharp(G) = \sup \mathrm{comp}_p(f)$, where the supremum is taken on all G-equivariant maps $f : G \to L^p$. It is a result by Naor and Peres (see [NP11, Thm. 9.1]) that $\alpha_p^\sharp$ is a QI-invariant among finitely generated amenable groups. The purpose of this Appendix is to use Carette's examples from Theorem 1 above, to show that, as might be expected, $\alpha_p^\sharp$ is not QI-invariant among all finitely generated groups.

Theorem 8. *For $1 \leq p \leq 2$, the vanishing of $\alpha_p^\sharp$ is not a QI-invariant.*

Proof. We use the two groups Γ and Λ from the proof of Theorem 1. We will show that, for $1 \leq p \leq 2$, we have $\alpha_p^\sharp(\Gamma) = \frac{1}{p}$ while $\alpha_p^\sharp(\Lambda) = 0$.

1) $\alpha_p^\sharp(\Lambda) = 0$ for $p \in [1,2]$. Indeed, assume by contradiction $\alpha_p^\sharp(\Lambda) > 0$ for some p. Then Λ admits a proper affine isometric action on L^p. By Corollary 6.23 of [CDH10] (using $1 \leq p \leq 2$), the group Λ has the Haagerup property, which is a contradiction.
2) For every $p \geq 1$, we have $\alpha_p^\sharp(\Gamma) = \max\{\frac{1}{p}, \frac{1}{2}\}$. This will follow from Theorem 6.3 in [CV], once we check the three assumptions. The first one is amenability of $\overline{\mathrm{hol}(\Gamma)}$, which clearly holds. The second is that $\overline{\mathrm{hol}(\Gamma)}$ should be co-compact in a closed, connected subgroup of $\mathrm{GL}_2(\mathbf{R})$: here the upper triangular subgroup of $\mathrm{SL}_2(\mathbf{R})$ does the job. The third one is that the inclusion $\mathbf{R}^2 \rtimes \mathrm{hol}(\Gamma) \to \mathbf{R}^2 \rtimes \overline{\mathrm{hol}(\Gamma)}$ should induce a quasi-isometry in restriction to $\mathbf{R}^2$, where $\mathrm{hol}(\Gamma)$ is endowed with the discrete topology in the first semi-direct product, and both semi-direct products are endowed with the word length associated to a compact generating subset. A sufficient condition for this to hold, is given by Proposition 2.5(b) of [CV]: it is enough that $\mathbf{R}^2$ is exponentially distorted in $\mathbf{R}^2 \rtimes \mathrm{hol}(\Gamma)$, i.e. $\limsup_{m\to\infty} \frac{|mv|_S}{\log m} < \infty$ for every vector $v \in \mathbf{R}^2$ (where $|.|_S$ is word length with respect to some compact generating set S). This holds because the matrix $H = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ lies in $\mathrm{hol}(\Gamma)$. □

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